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自動造市商機制與流動性提供者獎勵問題

A Study of AMM Mechanisms and Liquidity Provider Rewards

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Abstract

This thesis analyzes automated market makers (AMMs) in decentralized finance, focusing on the Uniswap protocol (V2 and V3) [1,2]. It formalizes the trading and liquidity provision mechanisms, constructs a price dynamic justified by empirical observation, investigates expected returns for liquidity providers compared with depositing assets to the bank, and presents numerical results.

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Chapter 1

Introduction

Automated market makers (AMMs) [3] are a novel trading mechanisms implemented on blockchain technology that has gained significant popularity recently. Unlike traditional financial market systems that rely on limit order books (LOBs) [4] to match buying and selling orders, AMMs facilitate trades by allowing participants to deposit assets into a pool, against which others can trade with the pool according to specific mathematical formulas. This innovative approach reduces computational loads, making AMMs suitable for blockchain implementation. Participants who contribute assets to these pools are known as liquidity providers (LPs) and earn trading fees as compensation.

This thesis focuses on Uniswap, the AMM with the highest trading volume, specifically its V2 and V3 versions. Uniswap V2 [1] operates under the constraint that the product of the amounts of each asset in the pool remains constant. In contrast, Uniswap V3 [2] introduces a more generalized concept of liquidity provision, allowing LPs to select specific price ranges within which to provide liquidity. This flexibility enhances capital efficiency by encouraging LPs to concentrate their assets in ranges where they anticipate price fluctuations.

This thesis begins by formulating the trading mechanism and liquidity provision in Uniswap V2 and V3, then constructing price dynamics based on arbitrage principles. We then analyze the growth rate of an LP's expected log reward. The LP's reward problem in V3 is modelled as an optimal stopping problem, where LPs decide when to withdraw their liquidity. Due to the difficulty of obtaining explicit results, we present a simplified strategy whose value function can be easily computed.

This thesis contributes to the understanding of AMMs in three ways. First, it provides a detailed theoretical formulation of the trading mechanisms in Uniswap V2 and V3. Second, it explores the implications of liquidity provision from the perspective of LPs, focusing on their expected returns and compute the numerical results for some simplified cases. Finally, it discusses potential extensions to the study. This work aims to offer a deeper understanding of the mechanisms underlying AMMs and their impact on liquidity providers' rewards by addressing these aspects.

Chapter 2

Uniswap V2

This chapter focuses on Uniswap V2 [1]. We introduce the fundamental mechanisms of Uniswap V2 and then construct a simplified model for price dynamics. Using this model, we derive the expected growth rate of a liquidity provider's (LP) wealth.

2.1 Uniswap V2 mechanism

Uniswap V2 is a decentralized exchange (DEX) that allows users to swap assets within a liquidity pool. Each pool contains two assets: a numéraire, X (typically a stablecoin pegged to the US dollar), and a risky asset, Y (usually a cryptocurrency). Traders can buy the risky asset from the pool by paying with the numéraire or sell it to the pool in exchange for the numéraire.

2.1.1 Pool reserves curve and price

Let (x_p, y_p) represent the reserves of assets X and Y in the pool. The Uniswap V2 constraint curve, denoted as $\Gamma_2(L)$, where $L > 0$ is the liquidity parameter, defines the relationship between these reserves:

$$\Gamma_2(L) \equiv \{(x_p, y_p) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid x_p \cdot y_p = L^2\} \tag{2.1}$$

This curve ensures the product of the reserves remains constant. The relationship between

x_p and y_p can also be expressed explicitly as:

$$x_p = \varphi_L(y_p) \equiv \frac{L^2}{y_p} \qquad \forall y_p \in \mathbb{R}^+ \tag{2.2}$$

where $\varphi_L(y_p)$ is a convex and strictly decreasing function.

2.1.2 Trading Mechanism and Price Impact

Given the constant product of reserves, a trader who wishes to buy $\Delta y \in (0, y_p]$ of asset Y from the pool must pay $\Delta x > 0$ of asset X, where Δx satisfies:

$$(x_p + \Delta x) \cdot (y_p - \Delta y) = L^2 \tag{2.3}$$

Conversely, a trader selling Δy of asset Y to the pool will receive Δx of asset X, where Δx satisfies:

$$(x_p - \Delta x) \cdot (y_p + \Delta y) = L^2 \tag{2.4}$$

The relative price of asset Y with respect to asset X is defined as the exchange rate when traders buy/sell an infinitesimal amount of asset Y from/to the pool. Using the notation in (2.2), this is:

$$Z \equiv -\varphi'_L(y_p) = \frac{x_p}{y_p} \tag{2.5}$$

Remark 2.1.1. The pool reserves pair can be determined by the current pool price Z and liquidity L:

$$(x_p, y_p) = R_2(L, Z) \equiv \left(L\sqrt{Z}, \frac{L}{\sqrt{Z}} \right) \in \Gamma_2(L) \tag{2.6}$$

This implies that the pool price and liquidity parameterize the reserves curve. ■

The pool price changes as trades occur, following (2.5). Buying asset Y adds X and removes Y from the pool, increasing the pool price. Selling Y does the opposite, decreasing the pool price. This is due to the convexity of the function ϕ_L (see Figure 2.1). The difference between the initial price and the actual trading price is called **price impact** or **slippage**. By increasing liquidity in the pool, price impact can be mitigated. We can express the pool price as a function

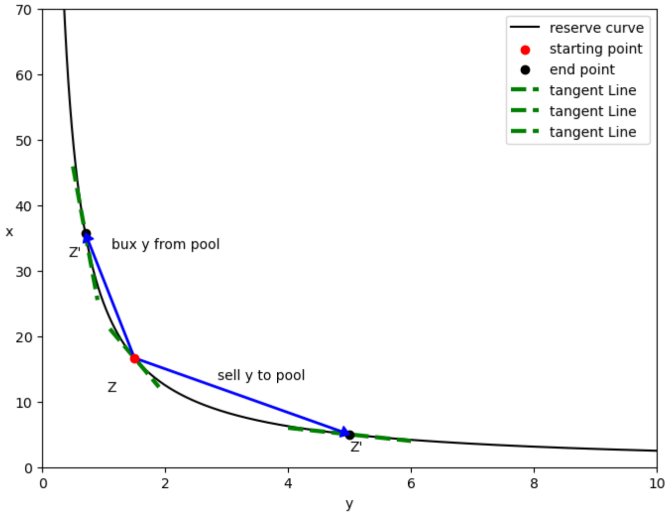


Figure 2.1: The red dot indicates the initial pool reserves pair. After trades, the reserves pair moves to one of the black dots (depending on buying or selling). Traders expect to trade at the current pool price Z (the slope of the green dashed line). However, the actual buying/selling price per unit of Y (the slope of the blue lines) is higher/lower. This illustrates price slippage: traders effectively buy/sell at a higher/lower price than expected. The new pool price Z is also higher/lower.

of asset Y reserves and compute the absolute value of its derivative:

$$Z(y_p) = \frac{L^2}{y_p^2} \Rightarrow \left|\frac{dZ(y_p)}{dy_p}\right| = \frac{2L^2}{y_p^3} = 2L^{-1}Z(y_p)^{\frac{3}{2}} \tag{2.7}$$

This shows that for a given pool price and trading volume, a pool with larger liquidity will experience less price change after a trade.

2.1.3 Liquidity Provision and Impermanent Loss

Besides traders, **liquidity providers (LPs)** play a crucial role in Uniswap V2. They deposit assets into the pool, ensuring sufficient reserves for trading. In return, LPs earn a share of the trading fees generated by the pool.

To maintain the current pool price Z when providing liquidity, an LP deposits amounts (x_{LP}, y_{LP}) of assets X and Y, respectively, satisfying:

$$\frac{x_p + x_{LP}}{y_p + y_{LP}} = Z \tag{2.8}$$

This ensures the ratio of assets in the pool remains consistent with the current price. The deposited amounts can be expressed in terms of the added liquidity $L' \equiv \sqrt{x_{LP} \cdot y_{LP}}$ and the current price:

$$(x_{LP}, y_{LP}) = R_2(L', Z) \tag{2.9}$$

The quantity L' represents the liquidity contributed by the LP, as the new pool reserves can be written as:

$$(x_p + x_{LP}, y_p + y_{LP}) = R_2(L + L', Z) \tag{2.10}$$

After depositing assets, the LP's share of the pool's assets is their **position**. LPs can withdraw any amount of liquidity in their position at any time which just reverses the process of liquidity provision. Figure 2.2 illustrates how changes in liquidity shift the reserves curve.

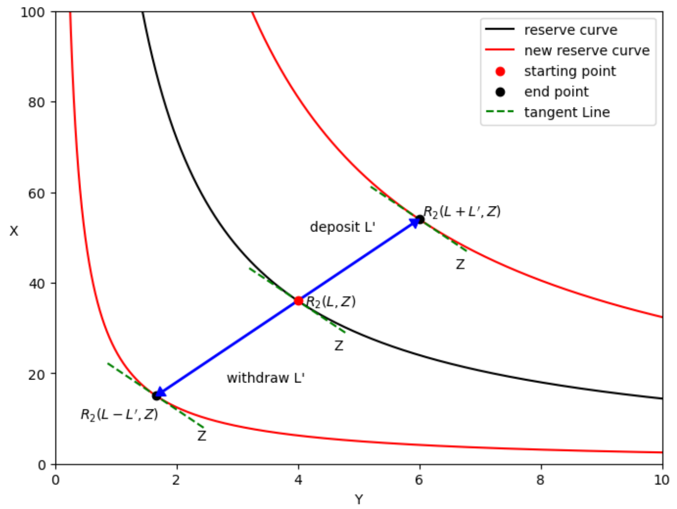


Figure 2.2: The red dot shows the initial pool reserves. Changes in liquidity shift the reserves curve (red curves). Pool reserves move to maintain the price (slope of tangent line).

A key risk for LPs is **impermanent loss** (IL). As the pool price Z changes, the value of the LP's position also changes, potentially leading to a loss compared to simply holding the assets at the beginning. Suppose the price changes to Z', and the LP's position becomes $(x'_{LP}, y'_{LP}) = R_2(L', Z')$. The difference in value compared to holding the assets can be quantified as:

$$\frac{(x'_{LP} + y'_{LP}Z') - (x_{LP} + y_{LP}Z')}{x_{LP} + y_{LP}Z'} = \frac{2\sqrt{\alpha}}{1+\alpha} - 1 \tag{2.11}$$

where $\alpha = \frac{Z'}{Z}$. This quantity is always non-positive, indicating that holding assets might be more profitable than providing liquidity.

2.1.4 Trading Fees

To incentivize liquidity provision and offset potential impermanent loss (IL), Uniswap V2 rewards liquidity providers (LPs) with a portion of the trading fees. The **fee constant** $\gamma \in (0, 1)$ determines the fraction of a trade collected as a fee.

For example, if a trader buys Δy of asset Y by paying Δx of asset X, the transaction modifies the reserves according to:

$$(x_p + \gamma\Delta x) \cdot (y_p - \Delta y) = L^2 \tag{2.12}$$

This equation shows that only $\gamma\Delta x$ of asset X is used for the actual trade, while the remaining $(1-\gamma)\Delta x$ is kept as a fee. An LP who has provided β percent of the liquidity to the pool will earn $\beta(1-\gamma)\Delta x$ of asset X as their share of the fee from this trade.

Similarly, for selling Δy of asset Y, the trader receives Δx of asset X, where:

$$(x_p - \Delta x) \cdot (y_p + \gamma\Delta y) = L^2 \tag{2.13}$$

Here, only $\gamma\Delta y$ of asset Y is exchanged, and the rest contributes to the fees.

If price fluctuations are not too large, the accumulated fees can compensate for IL. To illustrate, consider an LP providing one unit of liquidity when the pool price is initially 1. If the price increases to $\alpha > 1$, we can express the difference between the LP's wealth (including earned fees) and their wealth if they had held the assets in terms of α:

$$2\sqrt{\alpha} + \frac{1-\gamma}{\gamma}(\sqrt{\alpha} - 1) - (1 + \alpha) = -(\sqrt{\alpha} - 1)(\sqrt{\alpha} - 1 - \frac{1-\gamma}{\gamma}) \tag{2.14}$$

This expression is positive if and only if α falls within the range $\left(1, \frac{1}{\gamma^2}\right)$. This implies that fees can outweigh IL for moderate price increases. Similarly, if $\alpha < 1$, we can derive a region $(\gamma^2, 1)$ such that the fee can compensate IL.

In Uniswap V2, the fee is fixed at 0.3%, resulting in a range of approximately (0.994,1.006) where fees are expected to offset IL for small fluctuations.

2.2 Price Dynamics Under Arbitrage Opportunities

In this section, we present a simplified model for pool price dynamics based on the work in [3]. We make the following key assumption:

- There exists an external market, referred to as the reference market, that also trades assets X and Y. This market is assumed to have no trading cost and infinite liquidity, meaning trades do not affect the market price of asset Y relative to asset X.

The model's core concept is that significant deviations between the pool and reference market prices create **arbitrage opportunities**: the riskless exploitation of price differences to make a profit. This motivates traders to exploit these price discrepancies, driving the pool price towards a target value where no further arbitrage is possible. This process ensures the pool price closely tracks the reference market price, making the pool's valuation of asset Y reliable.

In the following, we will delve into the concept of arbitrage and use it to construct the price dynamics model.

2.2.1 Arbitrage

Traders who exploit price discrepancies for profit are called **arbitrageurs**. In this model, we assume arbitrageurs are the only traders in the pool, and any arbitrage opportunity is immediately and optimally exploited due to competition.

Let Z be the pool price and S the reference market price. Arbitrageurs profit by trading between the pool and the reference market. We denote the amounts of assets X and Y traded with the pool by $(\Delta x, \Delta y)$. If an arbitrageur buys Δy of asset Y from the pool with Δx of asset X and sells it on the reference market for $\Delta y \cdot S$ of asset X, his profit is:

$$-\Delta x + \Delta y \cdot S = \frac{1}{\Delta y}(S - Z_{\text{avg}}) \tag{2.15}$$

where $Z_{avg} = \frac{\Delta x}{\Delta y}$ is the average trading price with the pool, which is greater than or equal to Z due to slippage. Thus, if $Z < S$, there is a chance for arbitrage profit. Similarly, if $Z > S$, the arbitrageur might buy asset Y from the reference market and sell them to the pool for the potential profit.

To maximize profit, arbitrageurs must choose optimal Δx and Δy subject to the reserves curve constraint. Note that even if $Z \neq S$, arbitrage may not be possible due to trading fees, as explained below.

Since the pool price parameterizes the reserves curve, arbitrageurs effectively ”push” the pool price towards a target value after their trades. This determines the change in pool reserves and the corresponding trading volumes. The following proposition defines the optimal target prices under different scenarios:

Proposition 2.2.1. *The optimal pool price after arbitrage is:*

$$Z_{opt} = \begin{cases} \gamma^{-1}S & \text{if } Z > \gamma^{-1}S \\ Z & \text{if } Z \in [\gamma S, \gamma^{-1}S] \\ \gamma S & \text{if } Z < \gamma S \end{cases} \tag{2.16}$$

We say the prices S and Z satisfy the ***no-arbitrage condition*** *if $Z \in [\gamma S, \gamma^{-1}S]$ (or equivalently $S \in [\gamma Z, \gamma^{-1}Z]$).*

Proof. Let L be the pool's liquidity and $(x, y) = R_2(L, Z)$ the current pool reserves.

Case 1: $Z < S$ **(Pool price below market price)** In this case, the arbitrageur buys Δy of asset Y from the pool with Δx of asset X and sells the Y on the reference market. The arbitrageur's optimization problem is:

$$\max_{\Delta x, \Delta y \geq 0} -\Delta x + \Delta y \cdot S$$
$$\text{subject to } (x + \gamma\Delta x) \cdot (y - \Delta y) = L^2$$

We can reframe this problem in terms of the target pool price after the trade, Z'. Since buying Y increases the price, we have $Z' \geq Z$. The new reserves are $(x_{\text{new}}(Z'), y_{\text{new}}(Z')) = R_2(L, Z')$, and the trading volumes are $\Delta y = y - y_{\text{new}}(Z')$ and $\Delta x = \gamma^{-1}(x_{\text{new}}(Z') - x)$. Substituting these into the objective function and simplifying the problem becomes:

$$\min_{Z'\geq 0} g(x,y) \equiv \min_{Z'\geq 0} x + y\cdot\gamma S$$
$$\text{subject to} \begin{cases} x = x_{\text{new}}(Z') = L\sqrt{Z'} \\ y = y_{\text{new}}(Z') = L\frac{1}{\sqrt{Z'}} \\ Z' \geq Z \end{cases} \tag{2.17}$$

We can let $\phi(Z') = g\left(L\sqrt{Z'}, L\frac{1}{\sqrt{Z'}}\right)$ to incorporate the constraint and by chain rule we get:

$$\begin{aligned} \frac{d}{dZ'}\phi(Z') &= \frac{\partial g}{\partial x}\frac{dx}{Z'} + \frac{\partial g}{\partial y}\frac{dy}{Z'} \\ &= \frac{L}{2\sqrt{Z'}}\left(1 - \frac{\gamma S}{Z'}\right) \end{aligned} \tag{2.18}$$

So the function ϕ is strictly decreasing on $(0, \gamma S)$ and strictly increasing on $(\gamma S, \infty)$. We see that $\phi(Z')$ is strictly decreasing on $(0, \gamma S)$ and strictly increasing on $(\gamma S, \infty)$. The global minimum on $(0, \infty)$ is at $Z' = \gamma S$. However, due to the constraint $Z' \geq Z$, the optimal target price is:

$$Z_{\text{opt}} = \begin{cases} Z & \text{if } Z \geq \gamma S \\ \gamma S & \text{if } Z < \gamma S \end{cases} \tag{2.19}$$

Case 2: $Z > S$ **(Pool price above market price)** The arbitrageur sells Δy of asset Y bought from the reference market to the pool in exchange for Δx of asset X. The optimization problem is:

$$\max_{\Delta x, \Delta y \geq 0} \Delta x - \Delta y \cdot S$$
$$\text{subject to } (x - \Delta x)\cdot(y + \gamma\Delta y) = L^2$$

its equivalent form in terms of the target price Z' (where now $Z' \leq Z$) is:

$$\min_{Z' \geq 0} h(x, y) \equiv \min_{Z' \geq 0} x + y \cdot \gamma^{-1} S$$
$$\text{subject to} \quad \begin{cases} x = x_{\text{new}}(Z') = L\sqrt{Z'} \\ y = y_{\text{new}}(Z') = L\frac{1}{\sqrt{Z'}} \\ Z' \leq Z \end{cases} \tag{2.20}$$

Let $\psi(Z') = h\left(L\sqrt{Z'}, L\frac{1}{\sqrt{Z'}}\right)$ and compute the derivative:

$$\frac{d}{dZ'}\psi(Z') = \frac{L}{2\sqrt{Z'}}\left(1 - \frac{\gamma^{-1}S}{Z'}\right) \tag{2.21}$$

Following the same logic as in Case 1, we find:

$$\frac{d\psi(Z')}{dZ'} = \frac{L}{2\sqrt{Z'}}\left(1 - \frac{\gamma^{-1}S}{Z'}\right) \tag{2.22}$$

and the optimal target price is:

$$Z_{\text{opt}} = \begin{cases} Z & \text{if } Z < \gamma^{-1}S \\ \gamma^{-1}S & \text{if } Z \geq \gamma^{-1}S \end{cases} \tag{2.23}$$

Combining both cases, we obtain the desired result. □

Remark 2.2.2.

(1) The values $\gamma^{-1}Z$ in the no-arbitrage condition represent the infinitesimal buying price:

$$dx = \gamma^{-1}(\varphi_L(y - dy) - \varphi_L(y)) = -\gamma^{-1}\varphi_L'(y)dy + O\left((dy)^2\right) \sim \gamma^{-1}Zdy \tag{2.24}$$

Same for the infinitesimal selling price that equals to γZ. The optimal pool price ensures that these trading prices match the market price.

(2) If $\gamma = 1$ (no fees), the optimal price is simply $Z_{opt} = S$, and the pool price instantly

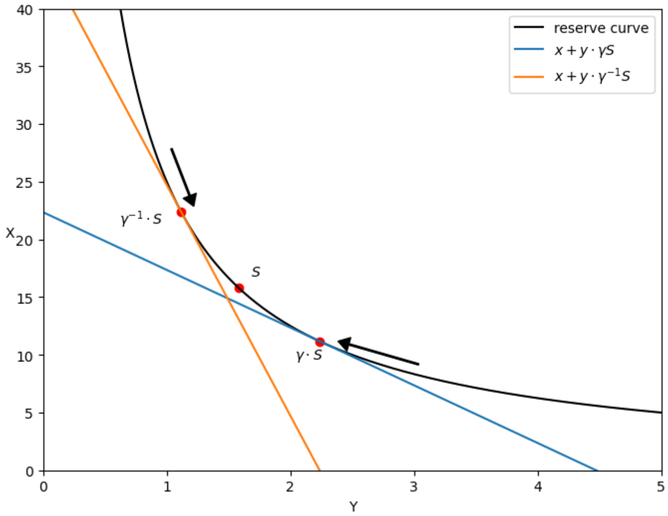


Figure 2.3: Arbitrage and Price Adjustment in Uniswap V2. The red dots indicate reserve pairs corresponding to different pool prices. If the pool price falls outside the no-arbitrage region $[\gamma S, \gamma^{-1}S]$, arbitrageurs will act to push the price back to the boundary points γS or $\gamma^{-1}S$, which are tangent to the reserves curve and represent the limits of profitable arbitrage.

matches the market price after any update. In this case, arbitrageurs essentially minimize the pool's mark-to-market wealth $X + Y \cdot S$.

(3) For $\gamma \neq 1$, arbitrageurs still minimize the mark-to-market value, but with effective market prices of $\gamma^{\pm 1}S$ due to fees. ■

2.2.2 Price Process

We now construct the pool price model based on arbitrage. We assume that the logarithm of the market price process, $\{\ln S_n\}_{n\geq 0}$, follows a simple random walk with forward probability $p > 0$ and step size $\delta > 0$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Given this market price process, arbitrage pushes the pool price to the target value whenever it falls outside the no-arbitrage region ,as defined in Equation (2.16). This leads to the following definition of the pool price process:

Definition 2.2.3. We define

- the **pool price process** $\{Z_n\}_{n\geq 0}$ as

$$Z_n = \begin{cases} Z_{n-1} & \text{if } \gamma S_n \leq Z_{n-1} \leq \gamma^{-1} S_n \\ \gamma S_n & \text{if } Z_{n-1} < \gamma S_n \\ \gamma^{-1} S_n & \text{if } Z_{n-1} > \gamma^{-1} S_n \end{cases} \tag{2.25}$$

with initial condition $Z_0 = S_0$. Here, $\mathbb{1}_A$ is the indicator function, equal to 1 if event A occurs and 0 otherwise.

- the **price deviation process** $\{M_n\}_{n\geq 0}$ as $M_n \equiv \ln S_n - \ln Z_n$

Since $Z_n \in [\gamma S_n, \gamma^{-1} S_n]$, we have $M_n \in [\ln \gamma, \ln \gamma^{-1}]$. To further simplify, we fix $k \in \mathbb{N}$ and set the random walk step size to $\delta = -k \ln \gamma$. This restricts the state space of M_n to $\{-k\delta, (-k+1)\delta, \ldots, (k-1)\delta, k\delta\}$, allowing us to show that $\{M_n\}_{n\geq 0}$ forms a Markov chain.

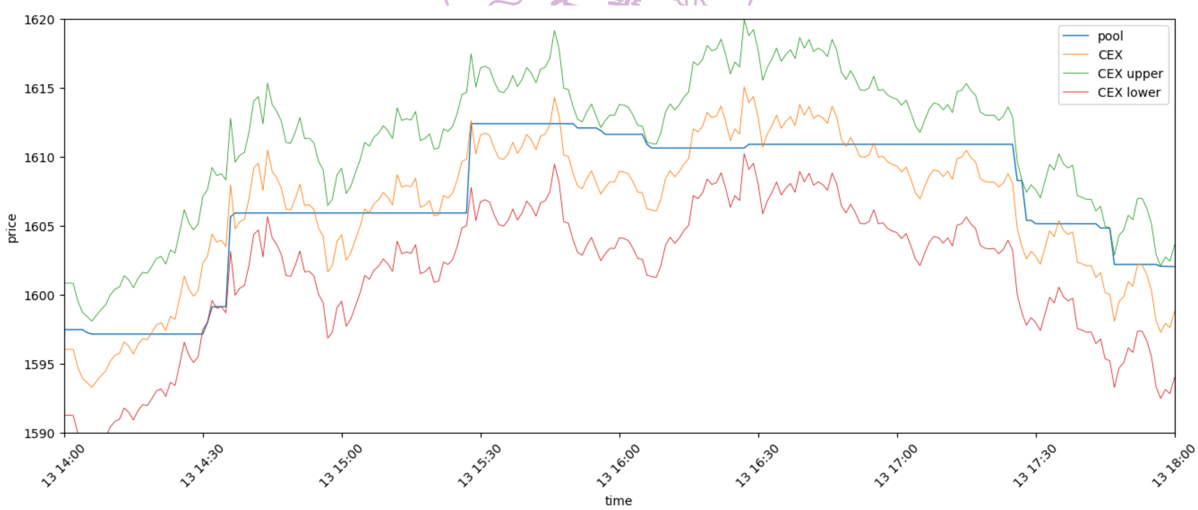


Figure 2.4: Pool vs. Market Price Trajectories. This graph shows the pool price (arbitrum WETH/USDC 0.3% pool) and market price (Binance ETH) from September 13, 2023, 2 PM to 6 PM. The pool price is bounded by the 'CEX upper' $\gamma^{-1}S$ and 'CEX lower' γS lines and is pushed towards them when touched, illustrating the arbitrage mechanism.

Proposition 2.2.4. *The process $\{M_n\}_{n\geq 0}$ is a Markov chain with an initial state $M_0 = 0$ and the following transition probabilities:*

For $m \neq \pm k$

$$\mathbb{P}(M_{n+1} = m'\delta \mid M_n = m\delta) = \begin{cases} p & \textit{if } m' = m+1 \\ 1-p & \textit{if } m' = m-1 \end{cases} \tag{2.26}$$

For $m = \pm k$

$$\mathbb{P}(M_{n+1} = m'\delta \mid M_n = k\delta) = \begin{cases} p & \textit{if } m' = k \\ 1-p & \textit{if } m' = k-1 \end{cases} \tag{2.27}$$

$$\mathbb{P}(M_{n+1} = m'\delta \mid M_n = -k\delta) = \begin{cases} 1-p & \textit{if } m' = -k \\ p & \textit{if } m' = -k+1 \end{cases} \tag{2.28}$$

Equivalently, this can be described by the transition matrix

$$\begin{array}{c} \\ -k \\ -k+1 \\ -k+2 \\ \vdots \\ k-2 \\ k-1 \\ k \end{array} \begin{array}{c} \begin{array}{cccccccc} -k & -k+1 & -k+2 & \cdots & k-2 & k-1 & k \end{array} \\ \begin{pmatrix} 1-p & p & 0 & \cdots & 0 & 0 & 0 \\ 1-p & 0 & p & \cdots & 0 & 0 & 0 \\ 0 & 1-p & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & p & 0 \\ 0 & 0 & 0 & \cdots & 1-p & 0 & p \\ 0 & 0 & 0 & \cdots & 0 & 1-p & p \end{pmatrix} \end{array} \tag{2.29}$$

Proof. Given $n \in \mathbb{N}$, from (2.25) it is easy to see that the step size of M_n is at most δ. We consider following cases:

Case 1. $M_n = m\delta \in (-k\delta, k\delta)$ **(Deviation within bounds)** . In this case, we have

$$\ln S_{n+1} - \ln Z_n = M_n + (\ln S_{n+1} - \ln S_n) = m\delta + (\ln S_{n+1} - \ln S_n)$$

Since the random walk step size is δ, this implies $-k\delta \leq \ln S_{n+1} - \ln Z_n \leq k\delta$. Rearranging, we get $Z_n \in [\gamma S_{n+1}, \gamma^{-1} S_{n+1}]$. Therefore, by Equation (2.25), we have $Z_{n+1} = Z_n$. This allows us to compute:

$$\begin{aligned}
\mathbb{P}(M_{n+1} = (m+1)\delta, M_n = m\delta) &= \mathbb{P}(\ln S_{n+1} - \ln Z_{n+1} = (m+1)\delta, \ln S_n - \ln Z_n = m\delta) \\
&= \mathbb{P}\left(\ln S_{n+1} - \ln Z_n = (m+1)\delta, \ln S_n - \ln Z_n = m\delta\right) \\
&= \mathbb{P}(\ln S_{n+1} - \ln S_n = \delta, M_n = m\delta) \\
&= p \cdot \mathbb{P}(M_n = m\delta)
\end{aligned}$$

where the last equality follows from the independence of random walk steps. Rearranging, we get:

$$\mathbb{P}(M_{n+1} = (m+1)\delta \mid M_n = m\delta) = p \tag{2.30}$$

Similarly, we can show that $\mathbb{P}\left(M_{n+1} = (m-1)\delta \mid M_n = m\delta\right) = 1 - p$.

Case 2. $M_n = k\delta$ **(Upper boundary)**. If $M_n = k\delta$, then $\ln S_{n+1} - \ln Z_n \geq (k-1)\delta$. From Equation (2.25), this implies $Z_{n+1} \geq Z_n$. We claim that:

$$\{M_{n+1} = k\delta, M_n = k\delta\} = \{\ln S_{n+1} = \ln S_n + \delta, M_n = k\delta\} \tag{2.31}$$

- LHS $\subseteq$ RHS: If the price deviation does not change and $\ln Z_{n+1} \geq \ln Z_n$, the market price must have increased, i.e., $\ln S_{n+1} = \ln S_n + \delta$.
- RHS $\subseteq$ LHS: If $\ln S_{n+1} = \ln S_n + \delta$ and $M_n = k\delta$, then $\ln S_{n+1} - \ln Z_n = (k+1)\delta > k\delta$. By Equation (2.25), this implies $\ln Z_{n+1} = \ln \gamma S_{n+1} = \ln S_{n+1} - k\delta$, and thus $M_{n+1} = k\delta$.

Therefore, the claim holds. Using the same reasoning as in Case 1, we find:

$$\mathbb{P}(M_{n+1} = k\delta \mid M_n = k\delta) = p \tag{2.32}$$

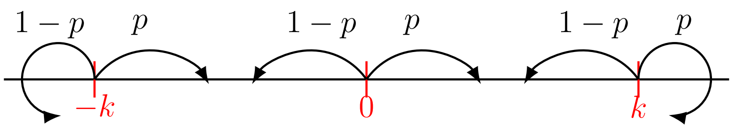
And since $M_{n+1} \leq k\delta$, we also have:

$$\mathbb{P}(M_{n+1} = (k-1)\delta \mid M_n = k\delta) = 1 - \mathbb{P}(M_{n+1} = k\delta \mid M_n = k\delta) = 1 - p \tag{2.33}$$

Case 3. $M_n = -k\delta$ **(Lower boundary)**. The argument is similar to Case 2, yielding the remaining transition probabilities. $\square$

Remark 2.2.5.

The Markov chain $\{M_n\}_{n\geq 0}$ behaves like a simple random walk when it is away from the boundaries $\pm k\delta$. However, it is "sticky" at the boundaries, remaining there if it tries to cross them. This can be visualized the following diagram:



$\blacksquare$

Since $\ln Z_n = \ln S_n - \ln M_n$, the increment of the logarithm of the pool price is also δ. We can further characterize the pool price process as follows:

Proposition 2.2.6. *For $n > 0$, define*

$$U_n \equiv \sum_{i=0}^{n-1} \mathbb{1}_{\{M_{n+1}=k\delta\ ,\ M_n=k\delta\}} \qquad D_n \equiv \sum_{i=0}^{n-1} \mathbb{1}_{\{M_{n+1}=-k\delta\ ,\ M_n=-k\delta\}} \tag{2.34}$$

Then we have

$$\ln Z_n = \ln S_0 + (U_n - D_n)\delta \tag{2.35}$$

Proof. It is sufficient to show that the following two statements hold:

$$\{\ln Z_{n+1} = \ln Z_n + \delta\} = \{M_{n+1} = k\delta, M_n = k\delta\} \tag{2.36}$$

$$\{\ln Z_{n+1} = \ln Z_n - \delta\} = \{M_{n+1} = -k\delta, M_n = -k\delta\} \tag{2.37}$$

i.e. the pool price moves if and only if the price deviation process remains at the boundaries. We will focus on proving Equation (2.36).

($\Rightarrow$) Assume $M_{n+1} = k\delta$ and $M_n = k\delta$. Using Equation (2.31), we get:

$$\ln Z_{n+1} - \ln Z_n = \ln S_{n+1} - \ln S_n - (M_{n+1} - M_n) = \delta$$

This implies $\ln Z_n < \ln \gamma S_{n+1}$. From the definition of the pool price process, Equation (2.25), we have $\ln Z_{n+1} = \ln Z_n + \delta$.

($\Leftarrow$) Assume $\ln Z_{n+1} = \ln Z_n + \delta$. From Equation (2.25), this can only happen if $\ln Z_{n+1} = \ln \gamma S_{n+1}$, as the other cases would contradict the fact that $\ln Z_{n+1} > \ln Z_n$. This implies $\ln Z_n = \ln S_{n+1} - (k+1)\delta$ and therefore $M_{n+1} = k\delta$. Now, suppose for contradiction that $M_n < k\delta$. Then,

$$\ln S_{n+1} - \ln S_n = \ln S_{n+1} - \ln Z_n - M_n > \delta$$

which contradicts the assumption that the random walk has step size δ. Therefore, we must have $M_n = k\delta$. This completes the proof of Equation (2.36). The proof of Equation (2.37) follows a similar logic. □

2.3 LP reward problem

Now we are interested in the growth rate of a LP's wealth plus the fee . A typical benchmark is depositing the numéraire to the bank earn the compound interest. If the growth rate is too slow, LP's position may be outperformed by the risk free rate, whose value grows exponentially fast due to the compound interest. So this section we focus on computing the asymptotic expected growth rate of the logarithm of LP's return (fee plus wealth).

Let (X_n, Y_n) be the LP's position in the pool at time n. We follow the argument in [5], where the author assumes that LP reinvest the fee back to the pool . Other than that, he does not withdraw or deposit liquidity after the initial deposition. The trading rule becomes the following:

$$\begin{cases} X_{n+1}^{\gamma} Y_{n+1} = X_n^{\gamma} Y_n & \text{if arbitrageur buys } Y \text{ from the pool} \\ X_{n+1} Y_{n+1}^{\gamma} = X_n Y_n^{\gamma} & \text{if arbitrageur sells } Y \text{ to the pool} \end{cases} \tag{2.38}$$

Then the fee are incorporated into the wealth of LP. It is shown that under the new trading rule, the pool price model still holds. Let $W_n = X_n + Y_n \cdot S_n$ denote the wealth of LP at time n. The result is the following :

Proposition 2.3.1.

$$\lim_{n\to\infty}\frac{\mathbb{E}\left[\ln W_n\right]}{n}=\begin{cases}\dfrac{\delta}{4k+2}\dfrac{1-\gamma}{1+\gamma} & \text{if} \quad p=\frac{1}{2}\\ \\ \dfrac{(2p-1)\delta}{1+\gamma}\dfrac{1-\gamma\rho^{2k+1}}{1-\rho^{2k+1}} & \text{if} \quad p\neq\dfrac{1}{2}\end{cases} \tag{2.39}$$

where $\rho=\frac{1-p}{p}$.

Before proving the proposition, we first show that $\{M_n\}_{n\geq 0}$ admits a stationary distribution, which will help us to study the limiting behavior.

Proposition 2.3.2. *The stationary distribution* $\pi:\{-k\delta,\cdots,k\delta\}\to\mathbb{R}^+$ *of the price deviation process* $\{M_n\}_{n\geq 0}$ *is given by the followings:*

(1) If $p=\frac{1}{2}$, *then* $\pi(m\delta)=\frac{1}{2k+1}$ *for all* $-k\leq m\leq k$.

(2) If $p\neq\frac{1}{2}$, *then* $\pi(m\delta)=\rho^{-(m+k)\delta}\frac{1-\rho}{\rho^{-2k}-\rho}$ *for all* $-k\leq m\leq k$.

Proof. Clearly, the Markov Chain is recurrent and irreducible, so the stationary distribution uniquely exists. For $p=\frac{1}{2}$, the transition matrix is doubly stochastic. Hence the Markov chain has a uniform stationary distribution π. For $p\neq\frac{1}{2}$, We just need to consider the linear equation $\boldsymbol{\pi}^T\mathbf{M}=\boldsymbol{\pi}$ where $\boldsymbol{\pi}=(\pi(-k\delta),\ldots,\pi(k\delta))^T$and $\mathbf{M}$ is the transition matrix given by (2.29). Solving this we can get

$$\pi(n\delta)=\rho^{-(n+k)}\pi(k\delta)\qquad\forall n\in\{-k\delta,\ldots,k\delta\}$$

and normalization condition gives the result. $\square$

Now we can prove Proposition 2.3.1:

Proof. For any $n \geq 0$, the logarithm of LP's mark-to-market wealth at time n can be written as

$$\begin{aligned} \ln W_n &= \ln(X_n + S_n Y_n) \\ &= \ln(X_n + Z_n Y_n) + \ln\left(\frac{X_n + S_n Y_n}{X_n + Z_n Y_n}\right) \\ &= \ln(X_n) + \ln 2 + \ln\left(\frac{1}{2} + \frac{S_n}{2Z_n}\right) \end{aligned} \tag{2.40}$$

Note that since $|M_n| = |\ln \frac{S_n}{Z_n}| \leq k$, the last term in the last equality is actually bounded:

$$\ln(\frac{1}{2} + \frac{S_n}{2Z_n}) \leq \ln(\frac{1}{2} \vee \frac{S_n}{2Z_n}) + \ln 2 = 0 \vee M_n$$

Hence it suffices to study the growth rate of the logarithm amount of X asset in LP's position. Under the arbitrage assumption and , we can rewrite (2.38) as

$$\begin{cases} X_{n+1}^{\gamma} Y_{n+1} = X_n^{\gamma} Y_n & \text{on } \{M_{n+1} = M_n = k\delta\} \\ X_{n+1} Y_{n+1}^{\gamma} = X_n Y_n^{\gamma} & \text{on } \{M_{n+1} = M_n = -k\delta\} \\ (X_{n+1}, Y_{n+1}) = (X_n, Y_n) & \text{otherwise} \end{cases} \tag{2.41}$$

Furthermore, from equations (2.36) and (2.37), we know how the pool price changes in the above cases, and therefore how the reserves change. Substituting $Y_n = \frac{X_n}{Z_n}$ into the above, simple calculation gives us

$$\frac{X_{n+1}}{X_n} = \begin{cases} e^{\frac{\delta}{1+\gamma}} & \text{on } \{M_{n+1} = M_n = k\delta\} \\ e^{-\frac{\gamma\delta}{1+\gamma}} & \text{on } \{M_{n+1} = M_n = -k\delta\} \\ 1 & \text{otherwise} \end{cases} \tag{2.42}$$

Now, for $n \geq 0$, we can express X_n in terms of $\{U_n\}_{n\geq 0}$ and $\{D_n\}_{n\geq 0}$:

$$X_n = e^{\frac{\delta}{1+\gamma}U_n - \frac{\gamma\delta}{1+\gamma}D_n} X_0 \Rightarrow \frac{\ln X_n}{n} = \frac{\ln X_0}{n} + \frac{U_n}{n}\frac{\delta}{1+\gamma} - \frac{D_n}{n}\frac{\gamma\delta}{1+\gamma} \tag{2.43}$$

By the ergodicity of $\{M_n\}_{n\geq 0}$, the limit of the above exists $\mathbb{P}$ a.s. (the state space is finite) and can be explicitly computed

$$
\begin{aligned}
\lim_{n\to\infty}\frac{U_n}{n} &= \mathbb{E}_\pi\left[\mathbb{1}_{\{M_1=M_0=k\delta\}}\right] \\
&= \pi(k\delta)\mathbb{P}(M_1 = k\delta \mid M_0 = k\delta) \\
&= \begin{cases} \dfrac{1}{4k+2} & \text{if } p = \dfrac{1}{2} \\ & \qquad\qquad \mathbb{P}\text{ - } a.s \\ p\rho^{-2k}\dfrac{1-\rho}{\rho^{-2k}-\rho} & \text{if } p \neq \dfrac{1}{2} \end{cases}
\end{aligned}
\tag{2.44}
$$

$$
\begin{aligned}
\lim_{n\to\infty}\frac{D_n}{n} &= \mathbb{E}_\pi\left[\mathbb{1}_{\{M_1=M_0=-k\delta\}}\right] \\
&= \pi(-k\delta)\mathbb{P}(M_1 = -k\delta \mid M_0 = -k\delta) \\
&= \begin{cases} \dfrac{1}{4k+2} & \text{if } p = \frac{1}{2} \\ & \qquad\qquad \mathbb{P}\ a.s \\ (1-p)\dfrac{1-\rho}{\rho^{-2k}-\rho} & \text{if } p \neq \frac{1}{2} \end{cases}
\end{aligned}
\tag{2.45}
$$

And the desired limit writes :

$$
\begin{aligned}
\lim_{n\to\infty}\frac{\ln W_n}{n} = \lim_{n\to\infty}\frac{\ln X_n}{n} &= \begin{cases} \dfrac{1}{4k+2}\dfrac{\delta}{1+\gamma} - \dfrac{1}{4k+2}\dfrac{\gamma\delta}{1+\gamma} & \text{if } p = \dfrac{1}{2} \\ & \qquad\qquad \mathbb{P}\text{ - } a.s \\ p\rho^{-2k}\dfrac{1-\rho}{\rho^{-2k}-\rho}\dfrac{\delta}{1+\gamma} - (1-p)\dfrac{1-\rho}{\rho^{-2k}-\rho}\dfrac{\gamma\delta}{1+\gamma} & \text{if } p \neq \dfrac{1}{2} \end{cases} \\
&= \begin{cases} \dfrac{\delta}{4k+2}\dfrac{1-\gamma}{1+\gamma} & \text{if } p = \dfrac{1}{2} \\ & \qquad\qquad \mathbb{P}\text{ - } a.s \\ \dfrac{(2p-1)\delta}{1+\gamma}\dfrac{1-\gamma\rho^{2k+1}}{1-\rho^{2k+1}} & \text{if } p \neq \dfrac{1}{2} \end{cases}
\end{aligned}
\tag{2.46}
$$

Since $|\ln X_n| \leq |\ln X_0| + n\delta$, by Lebesgue dominant theorem, we also have the convergence of the expected value. □

Remark 2.3.3. The limiting growth rate is positive if $p \geq \frac{1}{2}$. For $p < \frac{1}{2}$, we can see that the limiting growth rate is nonnegative if and only if

$$1 - \gamma\rho^{2k+1} \geq 0 \Rightarrow p \geq \frac{1}{1 + \gamma^{-\frac{1}{2k+1}}}$$

Still, whether the growth rate is greater than the interest rate or not depends on the choice of parameters. In 3.3.2 we will choose these parameters based on the empirical data and review the results derived here. ■

Chapter 3

Uniswap V3

In this section we introduce Uniswap V3, a more general model of Uniswap, which offers greater flexibility to LPs in their position. Similar to Chapter 2, We start from mechanism, construct price dynamics and then study LP's reward problem. Unlike in Uniswap V2, Uniswap V3 presents a more complex scenario due to its concentrated liquidity feature. So we will consider the stopping time problem for LP's reward.

3.1 Uniswap V3 mechanism

Uniswap V3 has more abstract pool structure compared to Uniswap V2. In this updated version, liquidity is not a constant but a step function of the pool price called **liquidity profile**:

$$L(z) \equiv \sum_{i \in \mathbb{Z}} L_i \mathbb{1}_{I_i}(z), \quad \forall\, z > 0,$$

where $I_i = [P_i, P_{i+1})$, is called a **tick interval** and P_i is called a **tick**. The union of tick intervals covers the whole the price range :

$$(0, \infty) = \bigcup_{i \in \mathbb{Z}} I_i$$

We assume that the liquidity is nonzero inside $[P_i, P_j)$ for some $i < j$ and is zero outside, which implies that the pool reserves curve is piecewise and Locally it looks the same as a reserves curve in Uniswap V2 but is truncated. Therefore, the model can be viewed as the patchwork of

Uniswap V2. Figure 3.1 and Figure 3.2 show an example of Uniswap V3.

The motivation of Uniswap V3 is to increase the capital efficiency and reduce slippage. In Uniswap V2, the liquidity is uniform on the whole price range, but in practice, only some small parts of liquidity are used because the pool price typically fluctuates within a limited range. On the contrary, in Uniswap V3, LPs can freely determine the tick intervals to provide liquidity instead of the whole price range, but only when the pool price is in their price ranges can they earn the trading fee . This encourages them to concentrate their liquidity in narrower price ranges where they anticipate the pool price would be. By doing so, they can earn more fee but with less initial assets to deposit. For traders, higher liquidity in the price range around the pool price can also reduce the slippage. We will give more detail argument in the later part.

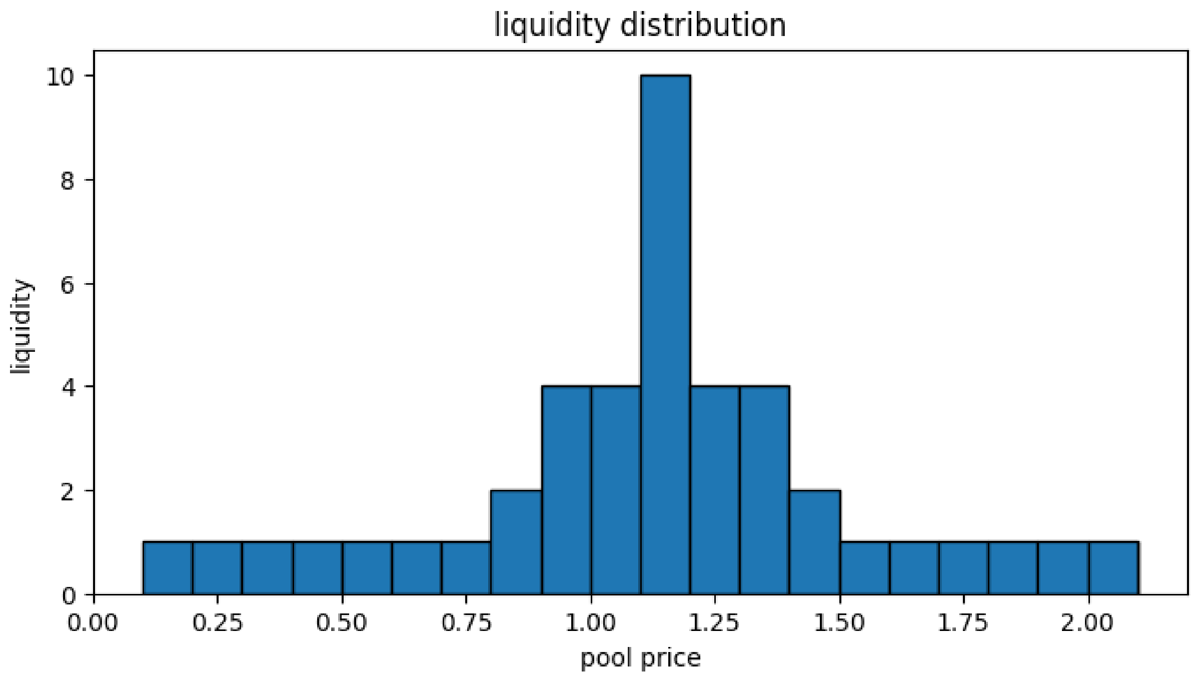


Figure 3.1: The graph gives an example of liquidity profile where the liquidity is concentrated in the range $[0.1, 2.1]$ and is zero outside.

3.1.1 Pool reserve curve and price

The pool reserves curve in Uniswap V3 can be considered as connecting multiple truncated Uniswap V2 reserves curves together. So we first define these small patches on all tick intervals and connect them together. Each tick interval I_i is just like a small pool that has its own reserves

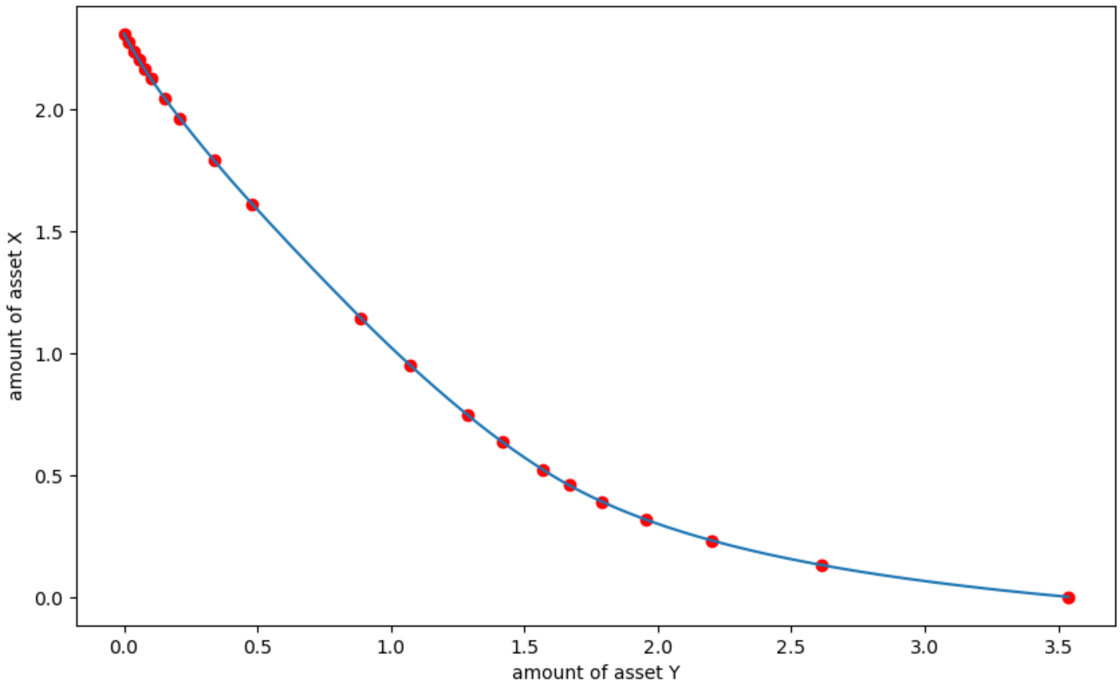


Figure 3.2: The pool reserves curve corresponding to Figure 3.1. Each red dot on the graph indicates a pool reserves pair that corresponds to some tick price. Equivalently, the absolute value of the slope of tangent line at the red dot is equal to some tick price. The curves between the adjacent red dots are actually truncated Uniswap V2 reserves curves but with different liquidity.

but with finite capacities x_i^* and y_i^* for asset X and Y. In Figure 3.2, these two quantities correspond to the difference of pool reserves between two adjacent red dots.

Definition 3.1.1. In Uniswap V3, the **reserves curve** on a tick interval $I_i = [P_i, P_{i+1})$ with liquidity $L_i \geq 0$ is defined by

$$\Gamma_3(I_i, L_i) \equiv \left\{ (x_i, y_i) \in [0, x_i^*] \times [0, y_i^*] \mid \left(x_i + L_i\sqrt{P_i}\right) \cdot \left(y_i + \frac{L_i}{\sqrt{P_{i+1}}}\right) = L_i^2 \right\} \tag{3.1}$$

where $x_i^* = L_i\left(\sqrt{P_{i+1}} - \sqrt{P_i}\right)$ and $y_i^* = L_i\left(\frac{1}{\sqrt{P_i}} - \frac{1}{\sqrt{P_{i+1}}}\right)$. We can also express the relation between the reserves pair (x_i, y_i) of the tick I_i as some explicit function:

$$x_i \equiv \phi_{I_i,L_i}(y_i) = \frac{L_i^2}{y_i + \frac{Li}{\sqrt{P_{i+1}}}} - L_i\sqrt{P_i} \tag{3.2}$$

for $y_i \in [0, y_i^*]$.

To connect the tick reserves curves together, we can parameterize them using a common parameter $Z > 0$:

$$\big(x_i(Z), y_i(Z)\big) = \begin{cases} (0, y_i^*) & \text{if } Z < P_i \\ \Big(L_i(\sqrt{Z} - \sqrt{P_i}), L_i(\frac{1}{\sqrt{Z}} - \frac{1}{\sqrt{P_{i+1}}})\Big) & \text{if } Z \in [P_i, P_{i+1}) \\ (x_i^*, 0) & \text{if } Z \geq P_{i+1} \end{cases} \tag{3.3}$$

for all $i \in \mathbb{Z}$. Then we see the relation between the parameter Z and the exchange rate of assets on each I_i:

$$\begin{aligned} Z_i &\equiv (Z \vee P_i) \wedge P_{i+1} \\ &= \frac{x_i(Z) + L_i\sqrt{P_i}}{y_i(Z) + \frac{L_i}{\sqrt{P_{i+1}}}} \\ &= -\phi'_{L_i,I_i}\big(y_i(Z)\big) \in [P_i, P_{i+1}] \end{aligned} \tag{3.4}$$

The pool reserves pair are actually the sum of reserves on each tick interval:

$$\big(x_p(Z), y_p(Z)\big) = \left(\sum_{i\in\mathbb{Z}} x_i(Z), \sum_{i\in\mathbb{Z}} y_i(Z)\right) = \left(\sum_{i\in\mathbb{Z}} L_i(\sqrt{Z_i} - \sqrt{P_i}), \sum_{i\in\mathbb{Z}} L_i(\frac{1}{\sqrt{Z_i}} - \frac{1}{\sqrt{P_{i+1}}})\right) \tag{3.5}$$

This is because whenever the parameter $Z \in I_k$, we can write:

$$(x_p, y_p) = \left(\sum_{i<k} x_i^* + x_k(Z) \ , \ \sum_{i>k} y_i^* + y_k(Z)\right) \tag{3.6}$$

which means locally the pool reserves change according to the rule we see in Uniswap V2. And the pool price, absolute value of slope of pool reserve curve's tangent line, is actually the parameter Z. ■

Remark 3.1.2. We can express x_i and y_i in the integral forms:

$$x_i = L_i(\sqrt{Z_i} - \sqrt{P_i}) = \int_{P_i}^{Z_i} \frac{1}{2} L(z) z^{-\frac{1}{2}} dz \tag{3.7}$$

$$y_i = L_i(\frac{1}{\sqrt{Z_i}} - \frac{1}{\sqrt{P_{i+1}}}) = \int_{Z_i}^{P_{i+1}} \frac{1}{2} L(z) z^{-\frac{3}{2}} dz \tag{3.8}$$

where $Z_i = (Z \vee P_i) \wedge P_{i+1}$. Therefore the pool reserves pair can be expressed as:

$$\begin{aligned}(x_p, y_p) &= \left(\sum_{i \in \mathbb{Z}} x_i \ , \ \sum_{i \in \mathbb{Z}} y_i \right) \\ &= \left(\int_0^Z \frac{1}{2} L(z) z^{-\frac{1}{2}} dz \ , \ \int_Z^\infty \frac{1}{2} L(z) z^{-\frac{3}{2}} dz \right)\end{aligned} \tag{3.9}$$

One can view $\frac{1}{2}L(z)z^{-\frac{1}{2}}$, $\frac{1}{2}L(z)z^{-\frac{3}{2}}$ as the **densities of assets X and Y**. ■

3.1.2 Trading Mechanism

The basic idea of trading mechanism in V3 is the same as in Uniswap V2: change of the pool reserves needs to follow the pool reserves curve. The key difference is that now the pool reserves curve is piecewise.

Suppose that a trader wants to buy $\Delta y > 0$ amount of asset Y with the current pool price Z. And the current pool reserves of asset Y is $y_p(Z)$. Then buying asset Y push pool price Z to the new pool price $\tilde{Z}$ that is given by:

$$\tilde{Z} \equiv \inf\{s > Z \mid y_p(Z) - y_p(s) = \Delta y \wedge y_p(Z)\} \tag{3.10}$$

and the pool reserves pair moves to the corresponding point $\left(x_p(\tilde{Z}), y_p(\tilde{Z})\right)$ on the curve. The amount of asset X the trader needs to pay is $\gamma^{-1}\left(x_p(\tilde{Z}) - x_p(Z)\right)$, due to the existence of fee, and receives $y_p(Z) - y_p(\tilde{Z})$ amount of asset Y. Note that if $\Delta y > y_p(Z)$, he can only receive $y_p(Z)$ amount of asset Y since this is all amount of assets Y in the pool. And the pool price is pushed to the rightmost tick that has nonzero liquidity. In practice, this scenario is unlikely since the pool price only fluctuates around the reference market price due to the arbitrage and the support of the liquidity profile is usually large enough to cover this range.

Similarly, if the trader sells Δy amount of asset Y with the current pool price Z. Selling pushes the price to the new pool price $\tilde{Z}$ given by:

$$\tilde{Z} \equiv \sup\{s < Z \mid x_p(s) - y_p(Z) = (\gamma \Delta y) \wedge (y^* - y_p(Z))\} \tag{3.11}$$

where $y^* \equiv \frac{1}{2}\int_0^\infty L(z)z^{-\frac{3}{2}}dz$ is the maximum pool capacity of asset Y.

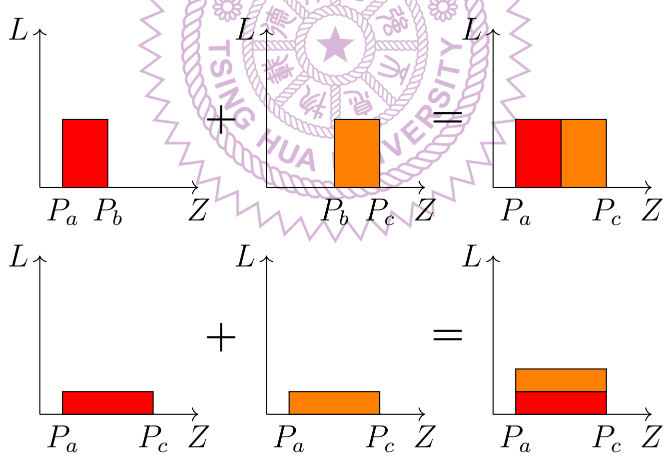
3.1.3 Liquidity provision and Trading fee

In Uniswap V3, LPs can freely choose the tick intervals and the amounts of liquidity they want to provide within those intervals, effectively building their own liquidity profile. And the pool liquidity profile is just the sum of all LPs' individual profiles.

Let $L_{LP}(\cdot)$ be a LP's liquidity profile and Z be the current pool price, the LP position can be expressed as

$$(x_{LP}(Z), y_{LP}(Z)) = \left(\int_0^Z \frac{1}{2}L_{LP}(\theta)\theta^{-\frac{1}{2}}d\theta, \int_Z^\infty \frac{1}{2}L_{LP}(\theta)\theta^{-\frac{3}{2}}d\theta\right) \tag{3.12}$$

An property of LP's position is **additivity**, which results from the linearity of the integration. This means the LP's position can be viewed as the sum of position on each tick interval, or the sum of positions on the same price range.



Uniswap V3 maintains the same principle as Uniswap V2 that trading fees are distributed to LPs based on their contribution to the pool. But now the contribution varies with price since it is evaluated by the portion of liquidity provide by LP.

Whenever a trade buys asset Y such that price moves from Z to $\tilde{Z}$, the trader pays

$$\frac{1-\gamma}{\gamma}\left(x_p(\tilde{Z}) - x_p(Z)\right) = \frac{1}{2}\int_Z^{\tilde{Z}\vee Z} \frac{1-\gamma}{\gamma} L_(\theta)\theta^{-\frac{1}{2}}d\theta$$

amount of asset X as fee. The corresponding amount earned by the LP, denoted by some function f_X, is given by

$$f_X(Z,\tilde{Z}) \equiv \frac{1}{2}\int_{Z}^{\tilde{Z}\vee Z} \frac{1-\gamma}{\gamma}\frac{L_{LP}(\theta)}{L(\theta)} \cdot L(\theta)\theta^{-\frac{1}{2}}d\theta = \frac{1}{2}\int_{Z}^{\tilde{Z}\vee Z} \frac{1-\gamma}{\gamma}L_{LP}(\theta)\theta^{-\frac{1}{2}}d\theta \tag{3.13}$$

Similarly, we use some function Y to denote the trading fee that is paid in asset Y to the LP given by:

$$f_Y(Z,\tilde{Z}) \equiv \frac{1}{2}\int_{\tilde{Z}\wedge Z}^{Z} \frac{1-\gamma}{\gamma}L_{LP}(\theta)\theta^{-\frac{3}{2}}d\theta \tag{3.14}$$

Note that if the LP does not provide the liquidity in the price range between Z and $\tilde{Z}$, then he has no contribution to the trade and the integrals vanish, i.e he can not earn any fee.

3.1.4 Advantage of Uniswap V3

As we mentioned earlier, Uniswap V3 can increase the capital efficiency. For example, suppose the pool price only fluctuates in some price range $[P_i, P_j]$ and there are two LPs A and B. The LP A deposits one unit of liquidity equally on $(0,\infty)$, so the amounts of assets he needs to deposit is

$$(x_A(Z), y_A(Z)) = \left(\sqrt{Z}, \frac{1}{\sqrt{Z}}\right) \tag{3.15}$$

where Z is the initial pool price. On the other hand, the LP B decides to deposit one liquidity unit of liquidity uniformly on $[P_i, P_j]$ and the amounts of assets he needs to deposit is

$$(x_B(Z), y_B(Z)) = \left(\frac{1}{2}\int_{P_i}^{Z} \theta^{-\frac{1}{2}}d\theta, \frac{1}{2}\int_{Z}^{P_{i+1}} \theta^{-\frac{3}{2}}d\theta\right) \tag{3.16}$$

We can see that A needs to deposit more asset X and asset Y than B does, but they receive the same amount of fee when the pool price remains in the region $[P_i, P_j]$, which suggest that concentrated liquidity profile is more capital efficient than uniform liquidity profile. Even if the pool price goes outside of the region, LPs can always adjust their liquidity profile to cover the new pool price.

3.2 Price Dynamics Under Arbitrage Opportunities

In this section, we will construct price dynamics for Uniswap V3, under the same arbitrage's assumption in Section 2.2. We will show that the result is same as in Uniswap V2 because whether the arbitrageur can earn the positive profit depends on the price, not the liquidity.

Let Z, S be the current pool and market price that are away from the leftmost and rightmost ticks of the price range having nonzero liquidity, we have

Proposition 3.2.1. *The optimal pool price for the arbitrageur is:*

$$Z_{opt} = \begin{cases} \gamma^{-1}S & \textit{if } Z > \gamma^{-1}S \\ Z & \textit{if } Z \in [\gamma S, \gamma^{-1}S] \\ \gamma S & \textit{if } Z < \gamma S \end{cases} \tag{3.17}$$

Proof. If $S \geq Z$, the arbitrageur determines a target pool price $\tilde{Z} \in [Z, \infty)$. He then buys $\Delta y \equiv y_p(Z) - y_p(\tilde{Z})$ amount of asset Y from the pool and pays $\Delta x \equiv \gamma^{-1}\left(x_p(\tilde{Z}), x_p(Z)\right)$. So the optimization problem for profit can be formulated as

$$\begin{aligned} &\sup_{\tilde{Z}\in[Z,\infty)} \left(y_p(Z) - y_p(\tilde{Z})\right) \cdot S - \gamma^{-1}\left(x_p(\tilde{Z}), x_p(Z)\right) \\ =& \sup_{\tilde{Z}\in[Z,\infty)} \frac{1}{2}\int_Z^{\tilde{Z}} L(\theta)[S\theta^{-\frac{3}{2}} - \gamma^{-1}\theta^{-\frac{1}{2}}]d\theta \\ =& \sup_{\tilde{Z}\in[Z,\infty)} \frac{1}{2}\int_Z^{\tilde{Z}} L(\theta)\theta^{-\frac{1}{2}}[S\theta^{-1} - \gamma^{-1}]d\theta \end{aligned} \tag{3.18}$$

Note that the integrand is positive if and only if $\theta < \gamma S$. If $Z > \gamma S$, the integrand will never be positive, the only choice is $Z_{opt} = Z$, i.e the arbitrageur does nothing. If $Z < \gamma S$, then clearly the optimal choice is $Z_{\text{opt}} = \gamma S$.

Similarly, if $S < Z$, the optimization problem for profit is:

$$\sup_{\tilde{Z}\in[0,Z]} \gamma^{-1}\left(y_p(Z) - y_p(\tilde{Z})\right)\cdot S + \left(x_p(Z) - x_p(\tilde{Z})\right) \tag{3.19}$$

$$= \sup_{\tilde{Z}\in[0,Z]} \frac{1}{2}\int_{\tilde{Z}}^{Z} L(\theta)[\theta^{-\frac{1}{2}} - S\gamma^{-1}\theta^{-\frac{3}{2}}]d\theta \tag{3.20}$$

$$= \sup_{\tilde{Z}\in[0,Z]} \frac{\gamma}{2}\int_{\tilde{Z}}^{Z} L(\theta)\theta^{-\frac{1}{2}}[\gamma - S\theta^{-1}]d\theta \tag{3.21}$$

which suggests that if $Z < \gamma^{-1}S$, then the optimal choice is $Z_{\text{opt}} = Z$. If not, we have $Z_{opt} = \gamma^{-1}S$. Combining the results we prove the statement. $\square$

Now we can consider the same construction of price dynamic as Section 2.2, where we define $\{S_n\}_{n\geq 0}$ to be a geometric random walk with step size $\delta = -\frac{\ln\gamma}{k}$ for some $k \in \mathbb{N}$ and the price deviation process $M_n \equiv \ln S_n - \ln Z_n$.

3.3 LP return problem

In the following discussion we focus on a simplified case that a LP only provide some constant liquidity $L > 0$ on some fixed price range $[P_a, P_b) = \bigcup_{i=a}^{b-1}[P_i, P_{i+1})$ that includes the initial pool price Z_0. Also, whenever the LP earns the trading fee, he deposits them into the bank to accumulate interest. We want to analyze the reward of the LP compared with depositing the initial wealth to the bank.

Based on Equation (3.12), his position can be expressed as

$$(x_{LP}(Z), y_{LP}(Z)) = \left(L(\sqrt{Z\wedge P_b} - \sqrt{P_a}), L(\frac{1}{\sqrt{Z\vee P_a}} - \frac{1}{\sqrt{P_b}})\right) \tag{3.22}$$

From the no-arbitrage condition that $Z_n \leq \gamma^{-1}S_n$, we can see that LP's wealth is actually bounded:

$$W_n \equiv x_{LP}(Z_n) + y_{LP}(Z_n)\cdot S_n \leq L(\sqrt{P_b} - \sqrt{P_a}) + \gamma^{-1}P_b\cdot L(\frac{1}{\sqrt{P_a}} - \frac{1}{\sqrt{P_b}}) \equiv W^* \tag{3.23}$$

Additionally, when the pool price leaves $[P_a, P_b]$, his position will be "locked" in the relative less valuable asset. For the fee part, if $p \neq \frac{1}{2}$, the SRW $\{\ln S_n\}_{n\geq 0}$ is transient, which means that eventually LP can not earn fee. For $p = \frac{1}{2}$, the SRW is null recurrent. Then let

$$F_N \equiv \sum_{n=0}^{N-1} f_X(Z_n, Z_{n+1}) + S_{n+1} f_Y(Z_n, Z_{n+1})$$

be the cumulative fee up to time N and

$$f_{max} \equiv \max\{f_X(P_a, P_b), \gamma^{-1} P_b f_Y(P_a, P_b)\} \tag{3.24}$$

which serves as an upper bound for the amount of fee earned for each price change. Then the asymptotic growth rate of the cumulative fee can be estimated by

$$\limsup_{N\to\infty} \frac{F_N}{N} \leq f_{max} \cdot \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{1}_{\{Z_n \in [P_a, P_b]\}} = 0 \tag{3.25}$$

since SRW is null recurrent. So the asymptotic growth rate is sublinear, which suggests that in the long run, the reward may be outperformed by our bench mark. This motivates us to consider the optimal stopping problem where and LP will choose an optimal timing to withdraw all his liquidity.

Unlike the case in section 2.3, to compute the exact wealth and the fee at each moment, only the price deviation is not enough. We also need the information of the pool price. Therefore, we consider the process of the pairs $\{(Z_n, M_n)\}_{n\geq 0}$, which is a Markov chain with transition probability:

$$\begin{aligned}
p\left((z_0, m_0), (z_1, m_0)\right) &= \begin{cases} p & \text{if} \quad m_0 = k\delta \ , \ z_1 = z_0 e^{\delta} \\ 1-p & \text{if} \quad m_0 = -k\delta \ , \ z_1 = z_0^{-\delta} \\ 0 & \text{else} \end{cases} \\
p\left((z_0, m_0), (z_0, m_1)\right) &= \begin{cases} p & \text{if} \quad m_0 \in (-k\delta, k\delta) \ , \ m_1 = m_0 + \delta \\ 1-p & \text{if} \quad m_0 \in (-k\delta, k\delta) \ , \ m_1 = m_0 - \delta \\ 0 & \text{else} \end{cases}
\end{aligned} \tag{3.26}$$

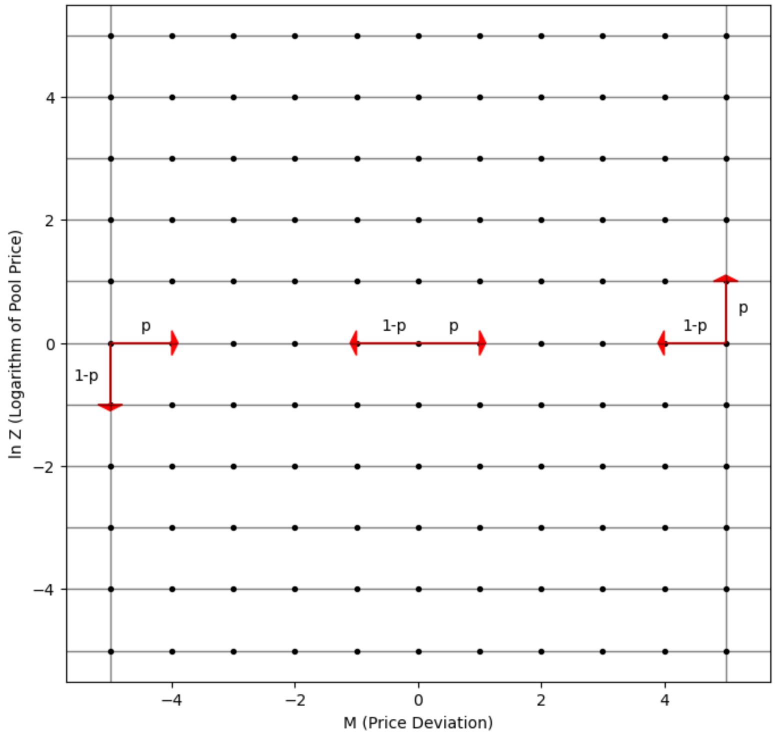


Figure 3.3: An example of the state space of $\{(lnZ_n, M_n)\}_{n\geq 0}$ with $k = 5$. The red arrow indicates the possible movement of the process at each state. We can see that the pool price can move to the next layer only when it is at the $M = \pm k$.

For simplicity, we further assume that ticks are given by $P_{i+1} = e^{\delta} P_i$. The state space of this Markov chain is visualized in Figure 3.3.

3.3.1 Optimal stopping problem

First we write down the optimal value function, which is the optimal expected discounted reward of LP. Given pool price Z and price deviation M, we use denote the wealth part of LP by:

$$W(Z, M) \equiv L(\sqrt{Z \vee P_b} - \sqrt{P_a}) + Ze^M L(\frac{1}{\sqrt{Z \wedge P_a}} - \frac{1}{\sqrt{P_b}}) \tag{3.27}$$

Then the value function writes:

$$V(Z, M) = \sup_{\tau \in \mathcal{A}} \mathbb{E}_{(Z,M)} \left\{ e^{-\tau r} W(Z_\tau, M_\tau) + \sum_{k=0}^{\tau-1} e^{-(k+1)r} \left[f_X(Z_k, Z_{k+1}) + Z_{k+1} e^{M_{k+1}} f_Y(Z_k, Z_{k+1}) \right] \right\} \tag{3.28}$$

where $\mathcal{A}$ is the collection of all admissible stopping times for the filtration $\{\mathcal{F}_n\}_{n\geq 0}$ and we use the convention that $\sum_{k=0}^{-1}$ is zero so that $\sum_{k=0}^{\tau-1}$ in the right hand side of the equation is well-defined. Note that even if $\tau = \infty$, the terms inside the expectation at the right hand side is well- defined due to the discount factor.

We aim to show that the value function V satisfies some recursion relation called **Bellman equation**(we will not solve it in this thesis).

Denote the expected fee earned at the next time step as follows:

$$c(Z, M) \equiv e^{-r}\mathbb{E}_{(Z,M)}\left[f_X(Z_0, Z_1) + Z_1 e^{M_1} f_Y(Z_0, Z_1)\right]$$

Our first step is to use the function c and the lemma below to rewrite the fee part in value function.

Lemma 3.3.1. *For $\tau \in \mathcal{A}$, we have:*

$$\mathbb{E}_{(Z,M)}\left[\sum_{k=0}^{\tau-1} e^{-(k+1)r} f_X(Z_k, Z_{k+1}) + Z_{k+1}e^{M_{k+1}} f_Y(Z_k, Z_{k+1})\right] = \mathbb{E}_{(Z,M)}\left[\sum_{k=0}^{\tau-1} e^{-kr} c(Z_k, M_k)\right] \tag{3.29}$$

Proof.

$$\begin{aligned}
&\mathbb{E}_{(Z,M)}\left[\sum_{k=0}^{\tau-1} e^{-kr} c(Z_k, M_k)\right] \\
=&\sum_{n\geq 1}\mathbb{E}_{(Z,M)}\left[\sum_{k=0}^{\tau-1} e^{-kr} c(Z_k, M_k); \tau = n\right] \\
=&\sum_{n\geq 1}\sum_{k=0}^{n-1}\mathbb{E}_{(Z,M)}\left[e^{-(k+1)r}\mathbb{E}_{(Z_k,M_k)}(f_X(Z_0, Z_1) + Z_1 e^{M_1} f_Y(Z_0, Z_1)); \tau = n\right] \\
=&\sum_{k\geq 0}\sum_{n>k}\mathbb{E}_{(Z,M)}\left[e^{-(k+1)r}\mathbb{E}_{(Z,M)}(f_X(Z_0, Z_1) + Z_1 e^{M_1} f_Y(Z_0, Z_1) \mid \mathcal{F}_k); \tau = n\right] \\
=&\sum_{k\geq 0}\mathbb{E}_{(Z,M)}\left\{e^{-(k+1)r}\mathbb{E}_{(Z,M)}\left[f_X(Z_k, Z_{k+1}) + Z_k e^{M_{k+1}} f_Y(Z_k, Z_{k+1})\mathbb{1}_{\{\tau>k\}} \mid \mathcal{F}_k\right]\right\} \\
=&\mathbb{E}_{(Z,M)}\left[\sum_{k=0}^{\tau-1} e^{-(k+1)r} f_X(Z_k, Z_{k+1}) + Z_{k+1}e^{M_{k+1}} f_Y(Z_k, Z_{k+1})\right]
\end{aligned} \tag{3.30}$$

where in the third line we change the order of sum and the fourth line use the Markov property.

□

Next, we want to apply a fundamental theorem for optimal stopping theory from [6], which states that if the reward function satisfies some conditions, we can derive the Bellman equation for optimal stopping problem.

Theorem 3.3.2. *Let $\{A_n\}_{n\geq 0}$ be a Markov chain with state space E, $\tilde{\mathcal{A}}$ be the collection of stopping time for the filtration generated by $\{A_n\}_{n\geq 0}$. Given a measurable function $G : E \to \mathbb{R}$ satisfying:*

1. $\mathbb{E}_s \left[\sup_{0\leq n\leq N}|G(A_n)|\right] < \infty \quad \forall\, N > 0$
2. $\lim_{n\to\infty} G(A_n)$ *exists* $\mathbb{P}_s$ *a.s.* $\forall\, N > 0$

Then the optimal value function of the optimal stopping problem: $V(s) \equiv \sup_{\tau\in\tilde{\mathcal{A}}} \mathbb{E}_s [G(A_\tau)]$ satisfies the Bellman equation:

$$V(s) = G(s) \vee \mathbb{E}_s [V(A_1)] \tag{3.31}$$

Furthermore, let $\tilde{\tau}^* \equiv \inf\{n \geq 0 \mid V(A_n) = G(A_N)\}$. If $\tilde{\tau}^* < \infty$ $\mathbb{P}_s - a.s.$ for all $s \in E$. Then it is the optimal stopping time.

In our case, the reward function depends on time(through discounted factors) and the path(through the fee part). To apply Theorem 3.3.3, we need to make some modification. The idea of modification also follows the discussion in [6], we creates a new process to absorb the time and path dependent parts and the corresponding optimal value function is the same as the one in our problem and we apply Theorem 3.3.2 to the optimal stopping problem based on this new process.

Theorem 3.3.3. *The optimal value function V satisfies the following equation:*

$$V(Z, M) = W(Z, M) \vee e^{-r} \left\{\mathbb{E}_{(Z,M)} [V(Z_1, M_1)]\right\} + c(Z, M) \tag{3.32}$$

Also, $\tau^* \equiv \inf\{n \geq 0 \mid V(Z_n, M_n) = W(Z_n, M_n)\}$ is the optimal stopping time, i.e.

$$V(Z, M) = \mathbb{E}_{(Z,M)} \left\{e^{-\tau^* r}W(Z_{\tau^*}, M_{\tau^*}) + \sum_{k=0}^{\tau^*-1} e^{-(k+1)r} \left[f_X(Z_k, Z_{k+1}) + S_{k+1}f_Y(Z_k, Z_{k+1})\right]\right\} \tag{3.33}$$

Proof. The whole proof is separated into four steps. In first three steps we incorporate the discount factor and cumulative fee into the process. In the last step we apply the theorem to the modified process and derive the result for our case.

step 1: Let $A_n = \{(Z_n, M_n)\}_{n\geq 0}$, $\mathcal{S} = \mathbb{R} \times \{-k, \dots, k\}$ be the state space, $\{\mathcal{F}_n\}_{n\geq 0}$ be the filtration generated by $\{A_n\}_{n\geq 0}$ and $\{u_k\}_{k\geq 1}$ be i.i.d. Bernoulli random variables that are independent of $\{\mathcal{F}_n\}_{n\geq 0}$ with $\mathbb{P}(u_i = 1) = 1 - e^{-r} = \mathbb{P}(u_i = 0) = e^{-r}$. Define a new process $\tilde{A}_n : \Omega \to \tilde{\mathcal{S}} \equiv \mathcal{S} \cup \{dead\}$ by:

$$\tilde{A}_n = \begin{cases} A_n & \text{on } \bigcap_{i=1}^n \{u_i = 0\} \\ dead & \text{on } \bigcup_{i=1}^n \{u_i = 1\} \end{cases}$$

and we define $\tilde{A}_\infty \equiv dead$. Clearly, $\{\tilde{A}_n\}_{n\geq 0}$ is still a Markov chain. For $s_0, s_1 \in \tilde{S}$, the transition probability is given by:

$$\tilde{p}(s_0, s_1) = \begin{cases} 1 & s_0, s_1 = dead \\ 0 & s_0 = dead \neq s_1 \\ e^{-r} p(s_0, s_1) & s_0, s_1 \neq dead \\ 1 - e^{-r} & s_1 = dead \neq s_0 \end{cases}$$

We extend the domain of W and c to $\tilde{S}$ by letting $W(dead) = c(dead) = 0$. Now observe that given integrable function $F : \tilde{S} \to \mathbb{R}$ such that $F(dead) = 0$, for all $s \in S$ we have

$$\begin{aligned} \mathbb{E}_s\left[F(\tilde{A}_n)\right] &= \int_{\tilde{S}} \tilde{p}(s, ds_1) \int_{\tilde{S}} \dots \int_{\tilde{S}} \tilde{p}(s_{n-1}, ds_n) F(s_n) \\ &= \int_S e^{-r} p(s, ds_1) \int_S \dots \int_S e^{-r} p(s_{n-1}, ds_n) F(s_n) \\ &= \mathbb{E}_s\left[e^{-nr} F(A_n)\right] \end{aligned} \tag{3.34}$$

Now we claim that $\mathbb{E}_s(e^{-\tau r} W(A_\tau)) = \mathbb{E}_s(W(\tilde{A}_\tau))$ for any stopping time $\tau \in \mathcal{A}$ and

$s \in S$. Given $n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathbb{E}_s\left[e^{-\tau r}W(A_\tau); \tau = n\right] &= \mathbb{P}(\bigcap_{i=1}^{n}\{u_i = 0\})\mathbb{E}_s\left[W(A_n); \tau = n\right] \\
&= \mathbb{E}_s\left[W(A_n); \tau = n, u_i = 0 \quad \forall i = 1 \dots n\right] \\
&= \mathbb{E}_s\left[W(\tilde{A}_n); \tau = n, \tilde{A}_n \neq dead\right] \\
&= \mathbb{E}_s\left[W(\tilde{A}_\tau); \tau = n\right]
\end{aligned} \tag{3.35}$$

where in the first equation we use independence of $\{u_i\}_{i=1}^{n}$ and $\{\mathcal{F}_n\}_{n\geq 0}$. Similarly, we also have

$$\begin{aligned}
\mathbb{E}_s\left[\sum_{k=0}^{\tau-1} e^{-kr}c(A_k)\right] &= \sum_{n\geq 0}\sum_{k<n}\mathbb{P}(\bigcap_{i=1}^{k}\{u_i = 0\})\mathbb{E}_s\left[c(A_n); \tau = n\right] \\
&= \sum_{n\geq 0}\sum_{k<n}\mathbb{E}_s\left[c(A_k); \tau = n, \tilde{A}_n \neq dead\right] \\
&= \mathbb{E}_s\left[\sum_{k=0}^{\tau-1} c(\tilde{A}_k)\right]
\end{aligned} \tag{3.36}$$

Combine together and take supremum we get

$$V(s) = \sup_{\tau\in\mathcal{A}}\mathbb{E}_s\left[W(\tilde{A}_\tau) + \sum_{k=0}^{\tau-1} c(\tilde{A}_k)\right] \tag{3.37}$$

step 2: Let $\{\tilde{\mathcal{F}}_n\}_{n\geq 0}$ be the filtration generated by $\{\tilde{A}_n\}_{n\geq 0}$ and $\tilde{\mathcal{A}}$ be the collection of all stopping time for $\{\tilde{\mathcal{F}}_n\}_{n\geq 0}$, we will show that the supremum above can be taken over all stopping times in $\tilde{\mathcal{A}}$ instead of $\mathcal{A}$. Define $\tau_d = \inf\{n \geq 0 \mid \tilde{A}_n = dead\}$, which is a stopping time for $\{\tilde{\mathcal{F}}_n\}_{n\geq 0}$. Notice that if $\tau \in \mathcal{A}$, then $\tau \wedge \tau_d \in \tilde{\mathcal{A}}$. This is because given $n \in \mathbb{N}$, there exists some $\mathcal{F}_n$-measurable function $g_n : \mathcal{S}^n \to \mathbb{R}$ such that $\mathbb{1}_{\{\tau\leq n\}} = g_n(A_1, \dots, A_n)$. Note that under $\{\tau_d > n\}$, we $A_i = \tilde{A}_i$ for $i \leq n$. So the indicator function $\mathbb{1}_{\{\tau\leq n\}\cap\{\tau_d > n\}} = g_n(\tilde{A}_1, \dots, \tilde{A}_n)\mathbb{1}_{\{\tau_d>n\}}$ is $\tilde{\mathcal{F}}_n$- measurable. Therefore we have

$$\{\tau \wedge \tau_d \leq n\} = (\{\tau \leq n\} \cap \{\tau_d > n\}) \cup \{\tau_d = n\} \in \tilde{\mathcal{F}}_n$$

and furthermore,

$$\mathbb{E}_s\left[W(\tilde{A}_\tau)+\sum_{k=0}^{\tau-1}c(\tilde{A}_k)\right]=\mathbb{E}_s\left[W(\tilde{A}_{\tau\wedge\tau_d})+\sum_{k=0}^{\tau\wedge\tau_d-1}c(\tilde{A}_k)\right]\leq\sup_{\tilde{\tau}\in\tilde{\mathcal{A}}}\mathbb{E}_s\left[W(\tilde{A}_{\tilde{\tau}})+\sum_{k=0}^{\tilde{\tau}-1}c(\tilde{A}_k)\right]$$

since $W(dead)=c(dead)=0$. Conversely, for each stopping time $\tilde{\tau}\in\tilde{\mathcal{A}}$, there exists $\tilde{\mathcal{F}}_n$-measurable function $g_n:\tilde{\mathcal{S}}^n\to\mathbb{R}$ for each $n\geq 0$ such that $\mathbb{1}_{\{\tilde{\tau}=n\}}=g_n(\tilde{A}_0,\ldots,\tilde{A}_n)$. We define

$$\tau=\begin{cases} n & \text{on} \quad \{g_n(A_0,\ldots,A_n)=1\}\\ \infty & \text{on} \quad \bigcap_{n=0}^{\infty}\{g_n(A_0,\ldots,A_n)=0\}\end{cases}$$

Clearly τ is a stopping time of $\{\mathcal{F}_n\}_{n\geq 0}$. Notice that $\tau\wedge\tau_d=\tilde{\tau}\wedge\tau_d$ since given any $\omega\in\Omega$, if $\tau(\omega)=n<\tau_d(\omega)$ for some $n\geq 0$, then

$$\mathbb{1}_{\{\tilde{\tau}=n\}}(\omega)=g_n(\tilde{A}_0(\omega),\ldots,\tilde{A}_n(\omega))=g_n(A_0(\omega),\ldots,A_n(\omega))=1 \tag{3.38}$$

so $\tilde{\tau}\wedge\tau_d(\omega)=n$. Since ω and n are arbitrary, the equality holds on $\{\tau<\tau_d\}$. Similarly, if $\tau(\omega)\geq\tau_d(\omega)=n$ for some $n\geq 0$, then

$$\mathbb{1}_{\{\tilde{\tau}=k\}}(\omega)=g_k\left(\tilde{A}_0(\omega),\ldots,\tilde{A}_k(\omega)\right)=g_k(A_0(\omega),\ldots,A_k(\omega))=0 \tag{3.39}$$

for $k<n$, which implies $\tilde{\tau}(\omega)\geq n$. So $\tilde{\tau}(\omega)\wedge\tau_d(\omega)=n$ and we can conclude that the equality holds on Ω. So given $s\in\mathcal{S}$, we have

$$\begin{aligned}\mathbb{E}_s\left[W(\tilde{A}_{\tilde{\tau}})+\sum_{k=0}^{\tilde{\tau}-1}c(\tilde{A}_k)\right]&=\mathbb{E}_s\left[W(\tilde{A}_{\tilde{\tau}\wedge\tau_d})+\sum_{k=0}^{\tilde{\tau}\wedge\tau_d-1}c(\tilde{A}_k)\right]\\&=\mathbb{E}_s\left[W(\tilde{A}_{\tau\wedge\tau_d})+\sum_{k=0}^{\tau\wedge\tau_d-1}c(\tilde{A}_k)\right]\\&\leq\sup_{\tau'\in\mathcal{A}}\mathbb{E}_s\left[W(\tilde{A}_{\tau'})+\sum_{k=0}^{\tau'-1}c(\tilde{A}_k)\right]\\&=V(s)\end{aligned} \tag{3.40}$$

Therefore

$$V(s) = \sup_{\tilde{\tau} \in \tilde{\mathcal{A}}} \mathbb{E}_s \left[W(\tilde{A}_{\tilde{\tau}}) + \sum_{k=0}^{\tilde{\tau}-1} c(\tilde{A}_k) \right] \tag{3.41}$$

step 3: Now we incorporate the cumulative fee part into the Markov process. Given $\tilde{I}_0 \in \mathbb{R}$ and $\tilde{I}_n = \tilde{I}_{n-1} + c(\tilde{A}_{n-1})$ for all $n \in \mathbb{N}$, we define another new Markov chain $\{(\tilde{A}_n, \tilde{I}_n)\}_{n \geq 0}$ with transition probability

$$\hat{p}\left((s_0, I), (s_1, I + c(s_0))\right) = \begin{cases} 1 & s_0 = s_1 = dead \\ 0 & s_0 = dead \neq s_1 \\ e^{-r} p(s_0, s_1) & s_0, s_1 \neq dead \\ 1 - e^{-r} & s_1 = dead \neq s_0 \end{cases}$$

for any $s_0, s_1 \in \tilde{\mathcal{S}}$ and $I \in \mathbb{R}_+$. Define $G : \tilde{\mathcal{S}} \times \mathbb{R}^+ \to \mathbb{R}$ by $G(s, I) = W(s) + I$ and let $\tilde{V}(s, I) \equiv \sup_{\tilde{\tau} \in \tilde{\mathcal{A}}} \mathbb{E}_{(s,I)} \left[G(\tilde{A}_{\tilde{\tau}}, \tilde{I}_{\tilde{\tau}}) \right]$, so we have $V(s) = \tilde{V}(s, 0) = \tilde{V}(s, I) - I$ for $s \in S$. To apply the Theorem 3.3.2 on $\tilde{V}$, it is required that for any $(s, I) \in \tilde{\mathcal{S}} \times \mathbb{R}^+$, the function G satisfies:

1. $\mathbb{E}_{(s,I)} \left[\sup_{0 \leq n \leq N} (G(\tilde{A}_n, \tilde{I}_n) \right] = \mathbb{E}_{(s,I)} \left[\sup_{0 \leq n \leq N} W(\tilde{A}_n) + \tilde{I}_n) \right] < \infty \ \ \forall \ N \geq 0.$
2. $\lim_{n \to \infty} G(\tilde{A}_n, \tilde{I}_n) = \lim_{n \to \infty} \left(W(\tilde{A}_n) + \tilde{I}_n \right)$ exists $\mathbb{P}_{(s,I)}$-a.s.

From the beginning of this section we see that wealth (3.23) and the amount of fee earned at each time (3.24) are bounded:

$$\mathbb{E}_{(x,I)} \left[\sup_{0 \leq n \leq N} W(\tilde{A}_n) + \tilde{I}_n) \right] \leq W^* + I + \mathbb{E}_{(x,I)} \left[\tilde{I}_N \right] \leq W^* + I + N f_{max} < \infty \tag{3.42}$$

On the other hand, since $\tau_d < \infty$ $\mathbb{P}_{(s,I)} - a.s$, the process will eventually be absorbed. Hence

$$W(\tilde{A}_n) + \tilde{I}_n \xrightarrow{n \to \infty} W(dead) + \tilde{I}_{\tau_d} = I + \sum_{n=0}^{\tau_d - 1} c(\tilde{A}_n) < \infty \quad \mathbb{P}_{(x,I)} - a.s. \tag{3.43}$$

The theorem gives us:

$$\tilde{V}(s, I) = G(s, I) \vee \mathbb{E}_{(s,I)} \left[\tilde{V}(\tilde{A}_1, \tilde{I}_1) \right] \tag{3.44}$$

and the optimal stopping time for the modified problem is given by

$$\tilde{\tau}^* \equiv \inf\{n \geq 0 \mid G(\tilde{A}_n, \tilde{I}_n) = \tilde{V}(\tilde{A}_n, \tilde{I}_n)\} \tag{3.45}$$

Note that since the dead state is an absorbing state and $G(dead, I) = I = \tilde{V}(dead, I)$, we can conclude $\tilde{\tau}^* \leq \tau_d < \infty$ $\mathbb{P}_{(s,I)}$-a.s. In particular, when $I = 0$ and $s \neq dead$, we get

$$\begin{aligned} V(s) &= \tilde{V}(s, 0) \\ &= W(s) \vee \mathbb{E}_{(s,0)} \left[\tilde{V}(\tilde{A}_1, \tilde{I}_1) \right] \\ &= W(s) \vee e^{-r} \left\{ \mathbb{E}_{(s,0)} \left[V(A_1) \right] \right\} + c(s) \end{aligned} \tag{3.46}$$

step 4: we check (3.33). First we claim that $\tau^* \wedge \tau_d = \tilde{\tau}^*$. If $\omega \in \{\tau^* < \tau_d\}$, then $\tilde{A}_{\tau^*}(\omega) = A_{\tau^*}(\omega)$. By the definition of τ^*, we have

$$G(A_{\tau^*}, \tilde{I}_{\tau^*})(\omega) = W(A_{\tau^*})(\omega) + \tilde{I}_{\tau^*}(\omega) = V(A_{\tau^*})(\omega) + \tilde{I}_{\tau^*}(\omega) = \tilde{V}(A_{\tau^*}, \tilde{I}_{\tau^*})(\omega) \tag{3.47}$$

in other words, $\tau^*(\omega)$ is the smallest time satisfying (3.45). So $\tilde{\tau}^*(\omega) = \tau^*(\omega)$. Conversely, if $\omega \in \{\tau^* \geq \tau_d\}$, we see that

$$W(A_k)(\omega) \neq V(A_k)(\omega) \Rightarrow G(A_k, \tilde{I}_k)(\omega) \neq V(A_k, \tilde{I}_k)(\omega) \tag{3.48}$$

for $k < \tau_d(\omega)$ and hence $\tilde{\tau}^*(\omega) = \tau_d(\omega)$.

Now for $s \neq dead$ we have

$$
\begin{aligned}
V(s) &= \tilde{V}(s,0) \\
&= \mathbb{E}_{(s,0)}\left[G(\tilde{A}_{\tilde{\tau}^*}, \tilde{I}_{\tilde{\tau}^*})\right] \\
&= \mathbb{E}_{(s,0)}\left[W(\tilde{A}_{\tilde{\tau}^*}) + \tilde{I}_{\tilde{\tau}^*}\right] \\
&= \mathbb{E}_{(s,0)}\left[W(\tilde{A}_{\tau^* \wedge \tau_d}) + \sum_{k=0}^{\tau^* \wedge \tau_d - 1} c(\tilde{A}_k)\right] \\
&= \mathbb{E}_{(s,0)}\left[W(\tilde{A}_{\tau^*}) + \sum_{k=0}^{\tau^* - 1} c(\tilde{A}_k)\right] \\
&= \mathbb{E}_{s}\left[e^{-\tau^* r} W(A_{\tau^*}) + \sum_{k=0}^{\tau^* - 1} e^{-kr} c(A_k)\right]
\end{aligned}
\tag{3.49}
$$

So τ^* is the optimal stopping time and we finish the proof.

$\square$

Although we have derived the Bellman equation for our problem, directly solving it is challenging due to the infinite horizon and the absence of clear boundary conditions. Therefore, we will not pursue a direct solution to this equation. Instead, we will explore some simplified stopping time strategy in the next part.

3.3.2 Stop at boundary

In this subsection, we consider a simplified strategy where the stopping time is set to be the time when the pool price reach some fixed price levels P_l or P_u. In this case, the value function, i.e expected discounted reward, is given by:

$$
V(Z,M) = \mathbb{E}_{(Z,M)}\left\{ e^{-rT} W(Z_T, M_T) + \sum_{n=0}^{T-1} e^{-r(n+1)} \left[f_X(Z_n, Z_{n+1}) + Z_{n+1} e^{M_{n+1}} f_Y(Z_n, Z_{n+1})\right] \right\}
\tag{3.50}
$$

for $(Z,M) \in \{P_l, \ldots, P_u\} \times \{-k\delta, \ldots, k\delta\} \cup \{(P_l, -k\delta), (P_u, k\delta)\}$ and the stopping time

$$
T \equiv \inf\{n \geq 0 \mid Z_n = P_l \text{ or } P_u\} = \inf\{n \geq 0 \mid (Z_n, M_n) = (P_l, k\delta) \text{ or } (P_u, -k\delta)\}
$$

Our goal is to compute the value function directly and produce some numerical results which may indicate us the optimal choice of P_l and P_u. We first prove a lemma which is crucial for the computation.

Lemma 3.3.4. *Let* $\tau \equiv \inf\{k \geq 0 \mid Z_n \neq Z_0\}$. *For* $(Z, M) \in (P_l, P_u) \times \{-k\delta, \ldots, k\delta\}$, *we have*

$$\begin{aligned} V(Z, M) = &\mathbb{E}_{(Z,M)}\left[e^{-\tau r}; Z_\tau = Ze^{\delta}\right]\left[V(Ze^{\delta}, k\delta) + f_X(Z, Ze^{\delta})\right] \\ &+ \mathbb{E}_{(Z,M)}\left[e^{-\tau r}; Z_\tau = Ze^{-\delta}\right]\left[V(Ze^{-\delta}, -k\delta) + f_X(Z, Ze^{-\delta})\right] \end{aligned} \tag{3.51}$$

Proof. First notice that $T \geq \tau \quad \mathbb{P}_{(Z,M)} - a.s.$ because for the process to reach the boundary points, the pool price must change. Then we can rewrite (3.50):

$$\begin{aligned} V(Z, M) = &\mathbb{E}_{(Z,M)}\left\{\sum_{n=0}^{\tau-1} e^{-r(n+1)}\left[f_X(Z_n, Z_{n+1}) + Z_{n+1}e^{M_{n+1}} f_Y(Z_n, Z_{n+1})\right]\right\} \\ &+ \mathbb{E}_{(Z,M)}\left\{e^{-Tr} W(Z_T, M_T) \sum_{n=\tau}^{T-1} e^{-r(n+1)}\left[f_X(Z_n, Z_{n+1}) + Z_{n+1}e^{M_{n+1}} f_Y(Z_n, Z_{n+1})\right]\right\} \end{aligned} \tag{3.52}$$

For $n < \tau$ part, since the pool price does not change until time τ, the LP can not earn fee. So the terms in the sum vanish except $n = \tau - 1$. For $\tau \leq n < T$, we introduce the shift operator on the canonical probability space as $\theta_\tau : \mathcal{S}^{\mathbb{Z}_+} \to \mathcal{S}^{\mathbb{Z}_+}$. So for any $\omega = (s_0, s_1, \ldots) \in \mathcal{S}^{\mathbb{Z}_+}$, we have

$$\theta_\tau(\omega) = (s_{\tau(\omega)}, s_{\tau(\omega)+1}, \ldots) \quad \text{and} \quad T \circ \theta_\tau(\omega) = T(\omega) - \tau(\omega) \tag{3.53}$$

and using the shift operator, we can rewrite the sum:

$$\begin{aligned} &\sum_{n=\tau(\omega)}^{T(\omega)-1} e^{-(n+1)r}\left[f_X(Z_n, Z_{n+1})(\omega) + Z_{n+1}(\omega)e^{M_{n+1}(\omega)} f_Y(Z_n, Z_{n+1})(\omega)\right] \\ &= e^{-\tau(\omega)r} \sum_{n=0}^{T(\omega)-\tau(\omega)-1} e^{-(n+1)r}\left[f_X(Z_n, Z_{n+1})(\theta_\tau(\omega)) + Z_{n+1}(\theta_\tau(\omega))e^{M_{n+1}(\theta_\tau(\omega))} f_Y(Z_n, Z_{n+1})(\theta_\tau(\omega))\right] \\ &= e^{-\tau(\omega)r}\left\{\sum_{n=0}^{T-1} e^{-(n+1)r}\left[f_X(Z_n, Z_{n+1}) + Z_{n+1}e^{M_{n+1}} f_Y(Z_n, Z_{n+1})\right]\right\} \circ \theta_\tau(\omega) \end{aligned} \tag{3.54}$$

Same for the wealth part:

$$e^{-T(\omega)r}W(Z_T, M_T)(\omega) = e^{-\tau r}[e^{-Tr}W(Z_T, M_T)] \circ \theta_\tau(\omega)$$

Combine the results together we get

$$\begin{aligned}
V(Z, M) &= \mathbb{E}_{(Z,M)}\Bigg\{e^{-\tau r}\left[f_X(Z_{\tau-1}, Z_\tau) + Z_\tau e^{M_\tau} f_Y(Z_{\tau-1}, Z_\tau)\right] \\
&+ e^{-\tau r}\left[e^{-Tr}W(Z_T, M_T)\sum_{n=0}^{T-1} e^{-(n+1)r}\left[f_X(Z_n, Z_{n+1}) + Z_{n+1}e^{M_{n+1}} f_Y(Z_n, Z_{n+1})\right]\right] \circ \theta_\tau\Bigg\} \\
&= \mathbb{E}_{(Z,M)}\left\{e^{-\tau r}\left[f_X(Z_{\tau-1}, Z_\tau) + Z_\tau e^{M_\tau} f_Y(Z_{\tau-1}, Z_\tau) + V(Z_\tau, M_\tau)\right]\right\} \\
&= \mathbb{E}_{(Z,M)}\left(e^{-\tau r}; Z_\tau = Z_0 e^{\delta}\right)\left[V(Ze^{\delta}, k\delta) + f_X(Z, Ze^{\delta})\right] \\
&+ \mathbb{E}_{(Z,M)}\left(e^{-\tau r}; Z_\tau = Z_0 e^{-\delta}\right)\left[V(Ze^{-\delta}, -k\delta) + f_X(Z, Ze^{-\delta})\right]
\end{aligned} \tag{3.55}$$

where in the second equality we use the strong Markov property and in the last equality we just separate the case that pool price goes up or down. □

Clearly, for the two boundary points $(P_l, -k\delta)$ and $(P_u, k\delta)$, the values are just the wealth part:

$$\begin{aligned}
V(P_u, k\delta) = W(P_u, k\delta) &= L(\sqrt{(P_b \wedge P_u) \vee P_a} - \sqrt{P_a}) + LP_u e^{k\delta}\left(\frac{1}{\sqrt{(P_b \wedge P_u) \vee P_a}} - \frac{1}{\sqrt{P_b}}\right) \\
V(P_l, -k\delta) = W(P_l, -k\delta) &= L(\sqrt{(P_b \wedge P_l) \vee P_a} - \sqrt{P_a}) + LP_l e^{-k\delta}\left(\frac{1}{\sqrt{(P_b \wedge P_l) \vee P_a}} - \frac{1}{\sqrt{P_b}}\right)
\end{aligned} \tag{3.56}$$

For the others, the lemma suggests that we first can solve some recursion relation for the values at the states $(Z, \pm k\delta)$ for $Z \in \{P_l, \dots, P_u\}$ and later compute the values at intermediate states. More precisely, let $\mathcal{B} = \{P_l, \dots, P_u\} \times \{-k\delta, k\delta\} \cup \{(P_l, -k\delta), (P_u, k\delta)\}$ and $N \in \mathbb{N}$ such that $\ln P_u - \ln P_l = (N+1)\delta$, we enumerate the states in B by some bijective map:

$$h(n) = \begin{cases} (P_l e^{n\delta}, -k\delta) & \text{if } 0 \le n \le N \\ (P_l e^{n-(N+1)\delta}, k\delta) & \text{if } N+1 \le n \le 2N+1 \end{cases} \tag{3.57}$$

Then for each $0 \le n \le 2N+1$, let $(Z, M) \equiv h(n)$. Figure 3.4 shows an example of enumerat-

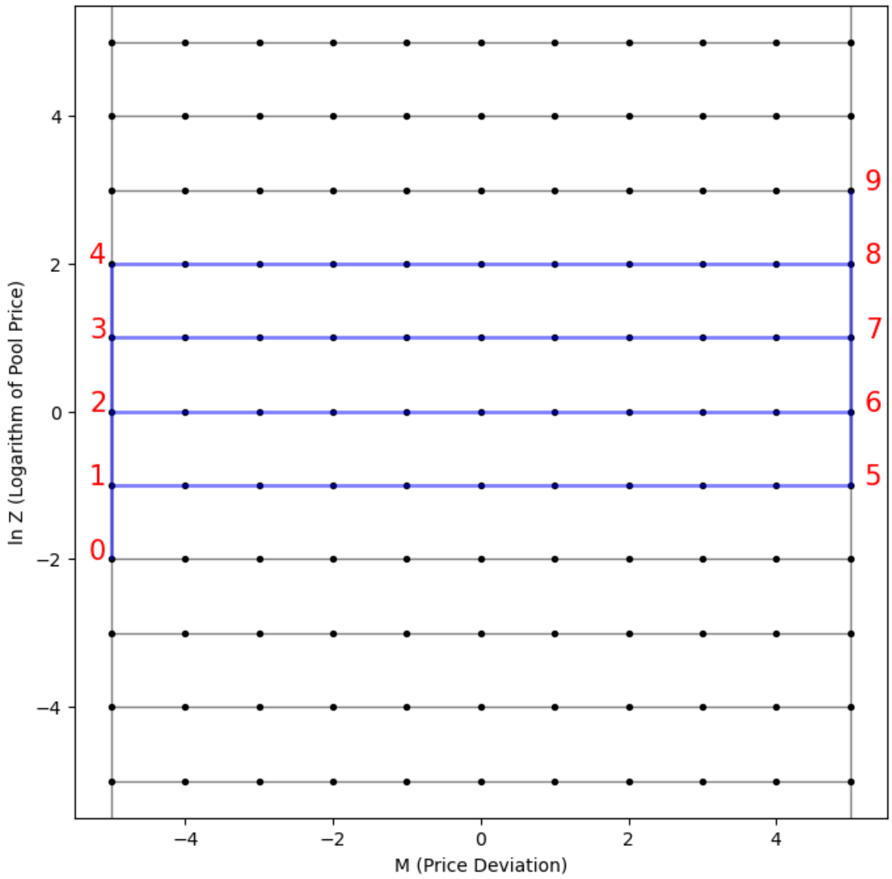


Figure 3.4: Enumerated states. An example of how h enumerates the boundary states. We can see the whichever enumerated state we start at, the first time pool price moves, we will reach another enumerated states. This implies that the value function on these states can be independently solve by Equation (3.51)

ing these boundary states. It suggests that we can use Equation (3.51) to solve the value function on these boundary states independently, without involving the intermediate state $m \neq \pm k\delta$. With the enumeration, we define the notations below:

- $W(n)$ corresponds to $W(Z, M)$
- $A_u(n)$ corresponds to $\mathbb{E}_{(Z,M)}\left[e^{-r\tau}; Z_\tau = Z_0 e^{\delta}\right]$
- $A_d(n)$ corresponds to $\mathbb{E}_{(Z,M)}\left[e^{-r\tau}; Z_\tau = Z_0 e^{-\delta}\right]$
- $S(n)$ corresponds to $Ze^{M\delta}$
- $f_X(n)$ corresponds to $f_X(Ze^{\delta}, Z)$
- $f_Y(n)$ corresponds to $f_Y(Ze^{-\delta}, Z)$

Substitute into (3.51), we derive the recursion relation for the value function on B:

$$
V(n) = \begin{cases} W(n) & \text{if } n = 0 \text{ or } 2N+1 \\ A_u(n)\left[V(n+N+1) + f_X(n)\right] + A_d(n)\left[V(n-1) + S(n-1)f_Y(n)\right] & \text{if } 0 < n \le N \\ A_d(n)\left[V(n+1) + f_X(n)\right] + A_d(n)\left[V(n-N) + S(n-N-1)f_Y(n)\right] & \text{if } N+1 \le n < 2N+1 \end{cases} \tag{3.58}
$$

We can also express recursion in linear system form. Use the notation $0 < n_1 \le N$ and $N+1 \le n_2 < 2N+1$, we can write:

$$
\begin{pmatrix} V(0) \\ \vdots \\ V(n_1) \\ \vdots \\ V(n_2) \\ \vdots \\ V(2N+1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 & 0 \\ \vdots & & & & & & \\ 0 & \dots & \overbrace{A_d(n_1)}^{n_1-1 \text{ th column}} & \dots & \overbrace{A_u(n_1)}^{n_1+N+1 \text{ th column}} & \dots & 0 \\ \vdots & & & & & & \\ 0 & \dots & \dots & \overbrace{A_d(n_2)}^{n_2-N \text{ th column}} & \dots & \overbrace{A_u(n_2)}^{n_2+1 \text{ th column}} & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V(0) \\ \vdots \\ V(n_1) \\ \vdots \\ V(n_2) \\ \vdots \\ V(2N+1) \end{pmatrix}
$$
$$
+ \begin{pmatrix} 0 \\ \vdots \\ A_u(n_1)f_X(n_1) + A_d(n_1)S(n_1-1)f_Y(n_1) \\ \vdots \\ A_u(n_2)f_X(n_2) + A_d(n_2)S(n_2-N-1)f_Y(n_2) \\ \vdots \\ 0 \end{pmatrix} \tag{3.59}
$$

To compute A_u and A_l, we first consider a general case. let $\{B_n\}_{n\ge0}$ be a simple random walk with forward probability $p > 0$ and unit step size. We want to compute $\mathbb{E}_x(e^{-T'r}\,;\, B_{T'} = k+1)$ and $\mathbb{E}_x(e^{-T'r}\,;\, B_{T'} = -(k+1))$ with $T' = \{n \ge 0 \mid B_n = k+1 \text{ or } -(k+1)\}$ and $x \notin \{-(k+1), (k+1)\}$. We can choose $a > 0$ such that $\{e^{aB_n - nr}\}_{n\ge0}$ is a martingale with respect to filtration generated by $\{B_n\}_{n\geq 0}$ and use the property of martingale to compute the desired expected value. More precisely, given $n > 0$, we want the following equation holds:

$$\mathbb{E}_x(e^{aB_{n+1}-(n+1)r} \mid B_n) = pe^{a(B_n+1)-(n+1)r} + (1-p)e^{a(B_n-1)-(n+1)r} = e^{aB_n - nr} \tag{3.60}$$

By some computation we get two solutions:

$$a_{\pm} = \log\left(\frac{e^r \pm \sqrt{e^{2r} - 4p(1-p)}}{2p}\right)$$

By the property of martingale, we get the linear equations for our desired expected values:

$$\begin{aligned} e^{a_{\pm}x} &= \mathbb{E}_x(e^{a_{\pm}X_0}) \\ &= \mathbb{E}_x(e^{a_{\pm}B_{T'} - T'r}) \\ &= \mathbb{E}_x(e^{a_{\pm}B_{T'} - T'r}\,;\, B_{T'} = k+1) + \mathbb{E}_x(e^{a_{\pm}B_{T'} - T'r}\,;\, B_{T'} = -k-1) \\ &= e^{(k+1)a_{\pm}}\,\mathbb{E}_x(e^{-T'r}\,;\, B_{T'} = k+1) + e^{-(k+1)a_{\pm}}\,\mathbb{E}_x(e^{-T'r}\,;\, B_{T'} = -k-1) \end{aligned} \tag{3.61}$$

Solve the equations and we derive

$$\begin{aligned} \mathbb{E}_x\left(e^{-rT'}\,;\, B_{T'} = k+1\right) &= \frac{e^{a_+(x+k+1)} - e^{a_-(x+k+1)}}{e^{2a_+(k+1)} - e^{2a_-(k+1)}} \\ \mathbb{E}_x\left(e^{-rT'}\,;\, B_{T'} = -k-1\right) &= \frac{e^{a_+(x-k-1)} - e^{a_-(x-k-1)}}{e^{-2a_+(k+1)} - e^{-2a_-(k+1)}} \end{aligned} \tag{3.62}$$

Now go back to our case, observe that $\{Z_\tau = Z_0 e^{\pm\delta}\} = \{M_n = M_{n+1} = \pm k\delta\}$. And since $\{M_n\}_{n\geq 0}$ behaves almost like a simple random walk except sticking at the states $\pm k\delta$ the event on the right hand side can actually be characterized by $\{B_{T'} = \pm(k+1)\}$. Therefore

$$\begin{aligned} \mathbb{E}_{(Z,M)}\left(e^{-r\tau}\,;\, Z_\tau = Z_0 e^{\delta}\right) = \mathbb{E}_{(Z,M)}\left(e^{-r\tau}\,;\, M_{\tau-1} = M_\tau = k\delta\right) &= \frac{e^{a_+(M+k+1)} - e^{a_-(M+k+1)}}{e^{2a_+(k+1)} - e^{2a_-(k+1)}} \\ \mathbb{E}_{(Z,M)}\left(e^{-r\tau}\,;\, Z_\tau = Z_0 e^{-\delta}\right) = \mathbb{E}_{(Z,M)}\left(e^{-r\tau}\,;\, M_{\tau-1} = M_\tau = -k\delta\right) &= \frac{e^{a_+(M-k-1)} - e^{a_-(M-k-1)}}{e^{-2a_+(k+1)} - e^{-2a_-(k+1)}} \end{aligned} \tag{3.63}$$

Notice that if $p = \frac{1}{2}$ and $r = 0$, we get $a = 0$ degenerate root but it is not a big problem since $\{B_n\}_{n\geq 0}$ is a martingale and the desired value can be derived by solving the gambler ruin

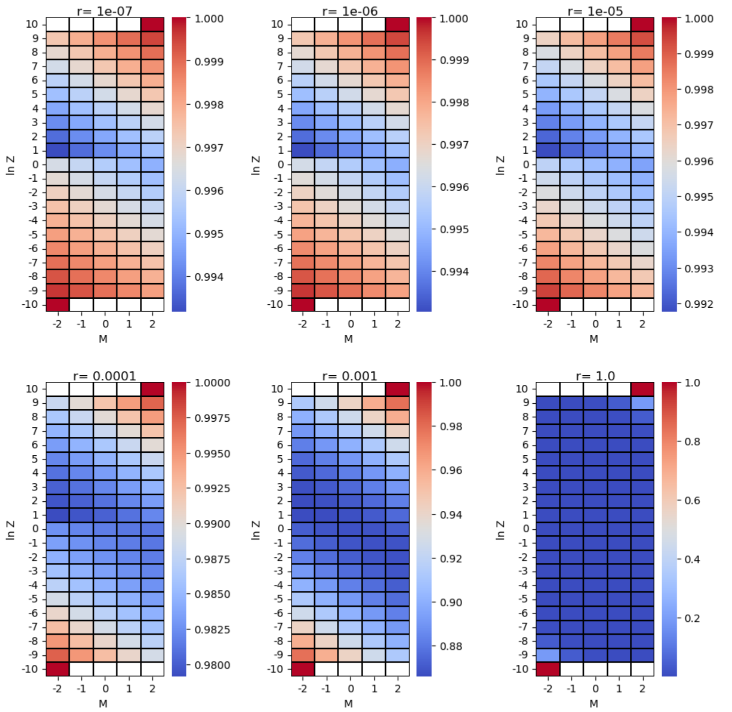


Figure 3.5: Normalized value functions with different interest rates. We use the parameters in Table 3.1 and divide the value functions by the initial wealth at states.

problem:

$$
\begin{aligned}
\mathbb{E}_{(Z,M)}\left(e^{-r\tau}\,;\,Z_\tau = Z_0 e^{\delta}\right) &= \mathbb{P}_{(Z,M)}(M_\tau = k+1) = \frac{M+k+1}{2(k+1)} \\
\mathbb{E}_{(Z,M)}\left(e^{-r\tau}\,;\,Z_\tau = Z_0 e^{-\delta}\right) &= \mathbb{P}_{(Z,M)}(M_\tau = -k-1) = \frac{k+1-M}{2(k+1)}
\end{aligned}
\tag{3.64}
$$

Substitute the above results into (3.51) and solve linear system for the value function at the states in $\mathcal{B}$. The values at intermediate states can be computed again by (3.51).

In the next parts, we present some numerical results to illustrate the value function and the optimal choice of the stopping boundaries for a simplified case. The time scale is set to 10 minutes and we choose Binance as our reference market. For the choice of parameters, the forward probability p and step size δ is chosen according to Binance ETH price 10 minutes data. We let p to be the empirical probability that the price goes up and let δ to be the mean of absolute value of log price change $|\ln S_{n+1} - \ln S_n|$. The corresponding k is $\lceil -\frac{1}{\delta} \ln \gamma \rceil$. And for simplicity, we choose the price range of the LP to be only the tick interval $[P_a, P_b) = [1, e^{\delta})$. The exact values of these parameters are list in Table 3.1 and Figure 3.5 shows the an example of value functions with different discounted rate , normalized by the initial wealth.

k	$\ln P_a$	$\ln P_b$	$\ln P_l$	$\ln P_u$	L	γ	δ	p
2	0	1	-10	10	1	0.997	0.0015	0.5

Table 3.1: Table of Parameters

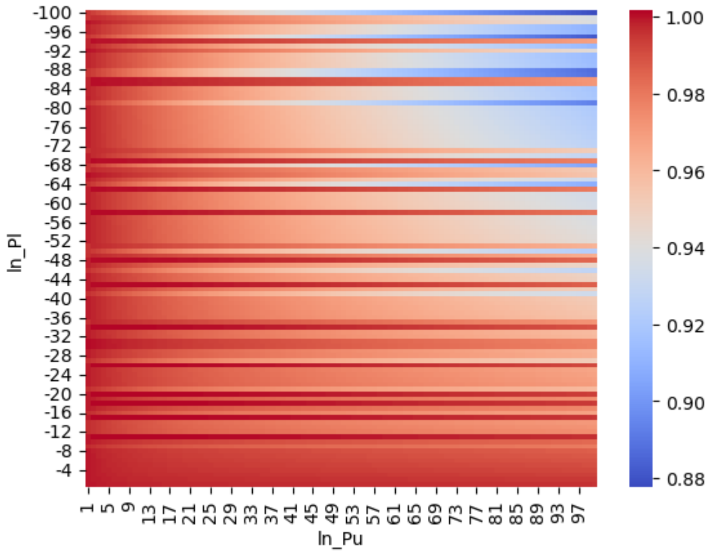


Figure 3.6: $V(0,0)/W(0,0)$ with different choice of $\ln P_l$ and $\ln P_u$. We can see there are horizontal stripes with significantly deeper colors, indicating that the values are higher at some specific choice of lower boundaries $\ln P_l$ under the same choice of upper boundary $\ln P_l$.

Next, we use the discounted rate r computed from 3% annual rate:

$$r = (1 - 0.03)^{\frac{1}{365 \cdot 24 \cdot 6}} - 1 \sim 5.6 \cdot 10^{-6} \tag{3.65}$$

and focus on the value at the starting point $(\ln Z_0, M_0) = (0, 0)$. Figure 3.6 compares the normalized value functions at $(0, 0)$ with different stopping boundaries $(\ln P_l, \ln P_u)$. We can derive the optimal choice of boundaries from $(-100, 100)$. The value is greater than 1, suggesting that the LP can have positive profit, see Table 3.2.

$\ln P_l$	$\ln P_u$	r	value
-18	10	$5.6 \cdot 10^{-6}$	1.001574840177328

Table 3.2: Table of discounted rate, optimal choice of stopping boundaries

We can also use the parameters in tables to compute the exact value of the asymptotic logarithm growth rate (2.39):

$$\lim_{n\to\infty}\frac{\ln W_n}{n}=\frac{\delta}{4k+2}\frac{1-\gamma}{1+\gamma}\sim 2.25\cdot 10^{-7}<r=5.6\cdot 10^{-6}$$

which means that in Uniswap V2, LP's reward is outperformed by bank deposit. While in Uniswap V3 we see that suitable choice upper and lower price levels help LP's reward to surpasses the bank deposit. So we can conclude that the generalization from V2 to V3 can indeed increase the profit of LPs by providing them more flexible choices to design their strategies.

3.3.3 Discussion

In this work, we only consider the stopping time as a control and compute the value function for stopping-at-boundary strategy. Though we did not solve the optimal stopping problem in section 3.3.1, the additivity property in Uniswap V3 simplifies consideration of a more general problem. Suppose now LP withdraws liquidity on each tick interval independently, which means that each tick interval is associated with an optimal stopping time. Since the decomposition of LP's position into smaller positions on tick intervals can also applies to wealth ,fee part and therefore the value function, we can focus on the optimal stopping problem for a position with unit liquidity on each tick interval only. Sum of the optimal value functions of these smaller positions gives the original optimal value function.

The results of the problem should provide some insight for more general control problem such as allowing LP to dynamically adjusting their liquidity profiles, since reducing the liquidity on a tick interval is actually equivalent to withdraw the position on that tick interval with the same amount of liquidity. Also advanced techniques like dynamic programming/ reinforcement learning can be employed to solve the problem more effectively.

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